THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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FEUERBACH'S THEOREM AS A COROLLARY.

BY PROF. E. H. NEVILLE, M.A.

THE solution of the problem of drawing a circle to touch a given line BC at a given point X and to touch also a given circle ν is very familiar; it is obvious that this solution must lead to an elementary verification of Feuerbach's theorem, but the extreme simplicity of this verification is commonly overlooked.

1. If the tangent at a point P of one circle is parallel to the tangent at a point Q of a second circle, and if the line PQ cuts the first circle again in P' and the second again in Q', then since the tangent at P' is parallel to the reflection of the tangent at P in PQ, and the tangent at Q' is parallel to the reflection of the tangent at Q in the same line, it follows that the tangent at P' is parallel to the tangent at Q'.

Applying this theorem, with P and Q coincident, we see that if a circle μ touches a line BC at X and touches a circle ν at O, the point in which the line XO cuts ν again is a point where the tangent is parallel to BC; applying the same theorem, but with P' and Q' coincident, we see that if R is a point of ν where the tangent is parallel to BC, and if RX cuts ν again in O, then the circle which touches BC at X and passes through O does touch ν at O.

This construction shows us that there are two and only two circles which touch BC at X and touch ν also, one circle corresponding to each end of the diameter of ν that is perpendicular to BC. To find whether a given circle which touches BC at X does touch ν, we have only to examine whether it is one of these two circles. For this purpose we can use a criterion involving the centre of the circle through X and O. If J is this centre and if L is the mid-point of XO, then JLR is a right angle, and therefore if V is the foot of the perpendicular from R on XJ,

$$XV \,.\, XJ = XL \,.\, XR = \tfrac{1}{2} XO \,.\, XR.$$

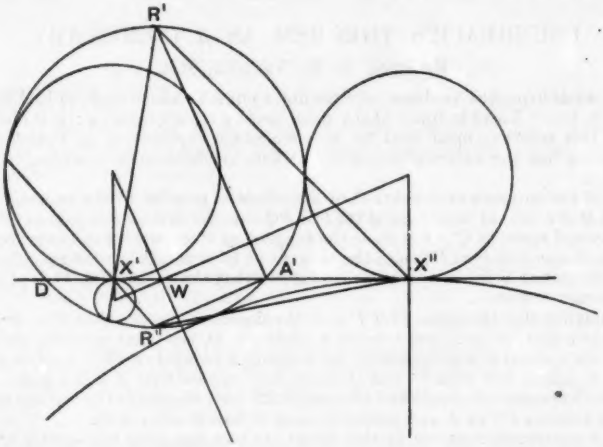
If U is the point in which the diameter of ν which is perpendicular to BC cuts BC, then $XV = UR$, and if we write Π for the power of X for the circle ν, the position of J for contact on XR is determined by the condition $UR \,.\, XJ = \frac{1}{2}\Pi$.

2. If ν is the nine-points circle of a triangle ABC, then ν cuts BC in two known points, namely, the midpoint A' of the side BC and the projection D of A on BC. The point U in which the diameter perpendicular to BC cuts BC is the midpoint of $A'D$ and is known at once, but the ends of the diameter are not among the nine sponsorial points, and a construction for them must be introduced. Since $B'C'$ is a chord perpendicular to this diameter, an end R bisects one of the two arcs into which B' and C' divide the circle, and therefore $A'R$ is one of the bisectors of the angles between $A'B'$ and $A'C'$. That

is to say, the ends of the diameter are the points in which the diameter is cut by the lines through A' parallel to the bisectors of the angle A of the triangle.

To see how this construction can be utilised, let R be joined to an arbitrary point T of BC, and let the line through J perpendicular to TR cut BC in W. Then since the triangles TUR, JXW are directly similar, $UR \cdot XJ = TU \cdot XW$, and the condition satisfied by J is equivalent to $TU \cdot XW = \frac{1}{2}\Pi$. When X and T are given, this condition determines W uniquely, and J is then found as the intersection of the line through X perpendicular to BC, with the line through W perpendicular to TR. It is interesting, though irrelevant, to notice that the condition which determines W does not discriminate between the two ends of the diameter of ν; if the lines joining T to the two ends are known, the centres of the two circles touching ν can be found from the one point W.

In view of the application to the nine-points circle, suppose now that the circle ν cuts BC in two points A', D, and take T at the first of these points.



Writing $\frac{1}{2}A'D$ for TU and $-A'X \cdot XD$ for the power Π, we have for the determination of W the condition $A'D \cdot WX = A'X \cdot XD$, which is equivalent to $A'D(A'X - A'W) = A'X(A'D - A'X)$, that is, to $A'D \cdot A'W = A'X^2$.

Let X be any point on a line BC which cuts a circle ν in two points A', D, and let R', R'' be the ends of the diameter of ν that is perpendicular to BC; let W be the point of BC which is such that $A'W$ is the third proportional to $A'D$ and $A'X$. Then the centre of the circle which touches BC at X and touches ν at a point on $R'X$ is on the line through W perpendicular to $A'R'$, and the centre of the circle which touches BC at X and touches ν at a point on $R''X$ is on the line through W perpendicular to $A'R''$.

If we think of W as given and of X as derived from W, we see that W must be on the same side of A' as D, and that if W satisfies this condition there are two possible positions X', X'' of X; these positions are determined jointly by the conditions that $X'X''$ has A' for its midpoint and that X', X'' are harmonic for D and W. Each of the lines through W determines a centre both of a circle which touches BC at X' and of a circle which touches BC at X'', and we can express the construction in such a way as to emphasise this point:

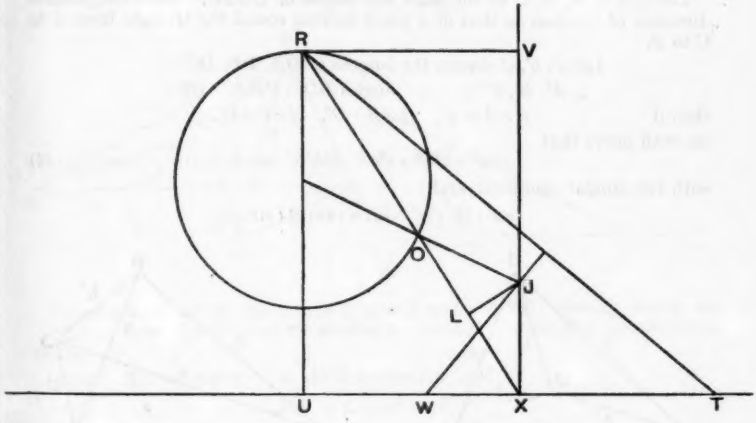
Let X', X'' and D, W be any two pairs of points harmonic with each other on a line BC, let A' be the midpoint of $X'X''$, and let R be either of the points in which

a circle ν through A' and D cuts the perpendicular bisector of $A'D$. Then the line through W perpendicular to $A'R$ contains both the centre of the circle which touches BC at X' and touches ν at a point on RX', and the centre of the circle which touches BC at X'' and touches ν at a point on RX''.

3. The verification of Feuerbach's theorem is immediate. Strictly speaking, all that we have to shew is that if I is any one of the contact centres, if X is the projection of I on BC, and if W is determined by the condition

$$A'D \cdot A'W = A'X^2,$$

then IW is parallel to one or other of the bisectors of the angle A. Actually, AI is itself one of these bisectors, and the verification consists in recognising



that the point in which AI cuts BC plays the part of W both for I and for the second contact centre on AI.

Let I', I'' be the two contact centres on a line which bisects the angle A and cuts BC in W, and let X', X'' be the projections of I', I'' on BC. Because $I'BI''$ and $I'CI''$ are right angles, the midpoint of $I'I''$ is the centre of a circle through B and C, and therefore the projection of this midpoint on BC, which is necessarily the midpoint of $X'X''$, is also the midpoint of BC. And because BI' and BI'' are the bisectors of the angle B, I' and I'' are harmonic for A and W, and therefore X' and X'' are harmonic for D and W. These two elementary properties of the points of contact X', X'' suffice not only to establish Feuerbach's theorem, but also to identify the four Feuerbach points.

Two features of this proof are interesting. The theorems of § 2 to which Feuerbach's theorem is a corollary are not lemmas introduced *ad hoc*, but simple results that would deserve notice even if no further inference was being drawn. And since in these theorems the two points of intersection A', D play quite dissimilar parts, there is an unusually complete failure if the circle does not in fact cut the line; the construction does not admit of modification to meet this case. But it is to be remarked that if the vertex A and the line BC are real, the points A' and D are real as well as the nine-points circle, even if the circumcircle does not cut BC; the construction remains possible in the most extravagant case which can be regarded as belonging to real geometry at all.

E. H. N.

CROSS RATIO AND A FOUR-POINT FIGURE.

By S. Barnard, M.A.

(1) The theorems stated below are immediate consequences of a simple equality, namely, that

$$If\ (z_1z_4,\ z_2z_3)=\rho,\ then\ (z_2z_4,\ z_3z_1)=1-\rho^{-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We use "$\angle BAC$" to denote the *least* angle through which AB must turn in order that it may lie along AC.

Thus $\angle CAB=-\angle BAC$, and we suppose that

$$-\pi<\angle BAC\leqslant\pi.$$

Let a, b, c, A, B, C be the sides and angles of $\triangle ABC$. Take the positive direction of rotation as that of a point moving round the triangle from A to C to B.

Let a', b', c' denote the lengths of DA, DB, DC;

„ A', B', C' „ „ angles BDC, CDA, ADB;

then if $\alpha=A+A',\quad \beta=B+B',\quad \gamma=C+C',$

we shall prove that

$$a^2a'^2=b^2b'^2+c^2c'^2-2bb'cc'\cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

with two similar equations, and

$$aa' : bb' : cc'=\sin\alpha : \sin\beta : \sin\gamma.$$

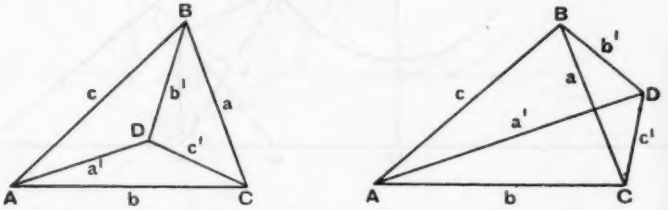


Fig. 1.

Let z_1, z_2, z_3, z_4 be complex numbers representing A, B, C, D, and let

$$(z_1z_4,\ z_2z_3)=\frac{z_1-z_2}{z_1-z_3}\cdot\frac{z_4-z_3}{z_4-z_2}=r(\cos\alpha'+i\sin\alpha').$$

Equating moduli and amplitudes,

$$r=cc'/bb',\ \alpha'=2m\pi+\angle CAB+\angle BDC=2m\pi+\alpha. \quad \ldots\ldots\ldots\ldots(3)$$

Changing the suffix 1 to 2, 2 to 3, 3 to 1, it follows that if

$$(z_2z_4,\ z_3z_1)=r'(\cos\beta'+i\sin\beta'),$$

then $r'=aa'/ac$ and $\beta'=2n\pi+\beta.$ $\ldots\ldots\ldots\ldots(4)$

Now by (3), (4) and the equality (1),

$$r'(\cos\beta+i\sin\beta)=1-r^{-1}(\cos\alpha-i\sin\alpha).$$

Therefore $r'^2=1-2r^{-1}\cos\alpha+r^{-2},$ $\ldots\ldots\ldots\ldots(5)$

and $\tan\beta=\sin\alpha/(r-\cos\alpha).$ $\ldots\ldots\ldots\ldots(6)$

Moreover, $\alpha+\beta+\gamma=\pm\pi.$ $\ldots\ldots\ldots\ldots(7)$

Whence easily

$$a^2a'^2=b^2b'^2+c^2c'^2-2bb'cc'\cos\alpha,$$

and $bb'\sin\gamma=cc'\sin\beta.$

Ex. 1. In the quadrilateral $ABDC$, given that the sum of the opposite angles A and D is α and that $AB \,.\, CD = CA \,.\, BD$, prove that

$$\angle ABC - \angle ADC = \tfrac{1}{2}(\pi - \alpha) \quad \text{and} \quad AD \,.\, BC = 2 \sin \frac{\alpha}{2} \,.\, AB \,.\, CD.$$

Ex. 2. If $(z_1 z_4, z_2 z_3)$ is real, then A, B, C, D are concyclic or collinear.

Ex. 3. If $(z_1 z_4, z_2 z_3) = -1$ and O is any point on the circle $ABCD$, then $O(AD, BC)$ is a harmonic pencil.

(2) Given three points A, B, C corresponding to complex numbers z_1, z_2, z_3, it is required to find a point D corresponding to z_4, such that

$$(z_1 z_4, z_2 z_3) = \rho = r(\cos \alpha + i \sin \alpha)$$

where ρ is a given complex number.

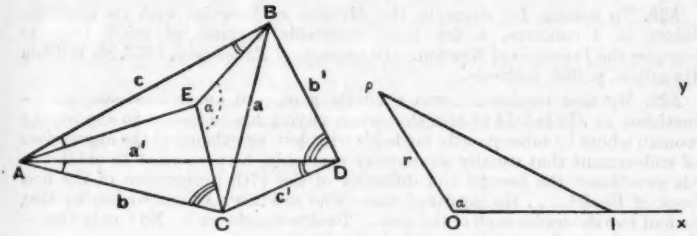


FIG. 2.

Construction. Mark the points 1, ρ and make $\triangle BEC$ directly similar to $\triangle \rho O1$. Draw $\triangle ACD$ directly similar to $\triangle AEB$. Then D is the required point.

Proof. The triangles ACE, ADB are similar, and

$$\angle BEC = \angle CAB + \angle BDC.$$

Also $BE/CE = cc'/bb'$, hence if z_4 corresponds to D and $(z_1 z_4, z_2 z_3) = \rho'$, we have

$$\text{am. } \rho' = 2m\pi + \angle CAB + \angle BDC = 2m\pi + \alpha,$$

and

$$|\rho'| = cc'/bb' = r;$$

therefore

$$\rho' = \rho.$$

(3) (i) Given three points A, B, C, it is required to find a point D such that A, B, C, D are equi-anharmonic (*i.e.* such that any cross ratio of A, B, C, D is equal to $-\omega$ or $-\omega^2$ where ω is an imaginary cube root of unity).

(ii) Show that there are two positions (D_1, D_2) of D and no more, and that for each of these

$$DA : DB : DC = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

(iii) If $\triangle ABC$ is named so that $A \leqslant 60°$, $B < 120°$, then the angles which BC, CA, AB subtend at D_1 are

$$A + 60°, \quad B + 60°, \quad C + 60° \quad \text{or} \quad 360° - (C + 60°),$$

and the angles which they subtend at D_2 are

$$60° - A, \quad \pm(60° - B), \quad \pm(60° - C).$$

That there are only two positions of D is due to the fact that if any cross ratio of A, B, C, D is $-\omega$ or $-\omega^2$, so also is each of the others.

Let $$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},$$

then $$\text{am.}(-\omega) = -\frac{\pi}{3}, \quad \text{am.}(-\omega^2) = \frac{\pi}{3},$$

and if in § 2 we take $\alpha = 2\pi - \frac{\pi}{3}$ or $\frac{\pi}{3}$, we have the following:

Construction. Draw the equilateral triangles BE_1C, BE_2C and make the triangles ACD_1, ACD_2 directly similar to AE_1B, AE_2B. Then D_1, D_2 are the required points. The rest follows from § 1.

S. BARNARD.

GLEANINGS FAR AND NEAR.

522. To master, for example, the *Minerva* of Sanctius with its commentators is, I conceive, a far more profitable exercise of mind than to conquer the *Principia* of Newton.—*Discussions of Philosophy*, 1853, Sir William Hamilton, p. 268, footnote.

523. My first teacher . . . was a kindly man, and an enthusiastic mathematician. . . . It is told of him that when paying his addresses to the worthy woman whom he subsequently made his wife, having exhausted the expressions of endearment that usually accompany courtship, he proceeded to exhibit to his sweetheart the beauty and difficulty of the 47th proposition of the first book of Euclid. . . . He mystified those who saw his six sons by saying that he had two sisters for each of the sons. Twelve daughters? No! only two.—J. Kerr, *Leaves from an Inspector's Logbook*.

524. Certainly I should not recommend mathematics as the remedy. Though one might naturally expect that the fault of mere mathematicians would be an over-rigid demand for demonstration in all subjects, I have found the fact to be the reverse. They generally, when they come to any other subject, throw off all regard to order and accuracy, . . . With them, mathematics is "Attention!" and everything else "Stand at ease!" The defect of mathematics as an exclusive or too predominant study is, that it has no connexion with human affairs, and affords no exercise of judgment, having no degrees of probability.—*Life of Whately*, 1868, ii. p. 211.

525. Whatever may be thought of the endeavour to exercise the discrimination of which geometry points out the possibility by framing or arguing on definitions, we do not remember to have seen one so well calculated for the mere beginner as the following: "A straight line is a straight line."—De Morgan, *Straight Line*, P.C. 23, p. 101.

526. Hume in a note added shortly before his death to the third appendix in his *History*: I suppose there is no mathematical, still less an arithmetical, demonstration that the road to the Holy Land was not the road to Paradise, as there is that the endless increase of national debts is the direct road to national ruin.

527. Christopher Smart, the poet, used to relate "that the first conversation with him (Johnson) was of such variety and length that it began with poetry and ended at fluxions."

528. We have been desired by a Person of Honour to publish the following . . . [Then it states that the Duke of Brunswick had erected an academy near his castle at Wolfenbuttle]: The first care and design is to give an Insight into the Principles of Religion; next to instruct in Eloquence, Politicks, Morality, History, Geography and other Parts of Mathematics. . . .—*Post Boy*, No. 2132, Jan. 11-13, 1708-9.

529. George Spencer used to teach mathematics to Mozley's brother.

SOME NEW CIRCLES CONNECTED WITH A TRIANGLE.

By H. V. Lowry, M.A.

It is sometimes fascinating to return to the old geometry and try to find new properties of the triangle and quadrilateral. One of the most interesting ways of discovering such properties is to start with some well-known result and generalise it. The following notes are the result of an attempt to find circles in a triangle which satisfy the same kind of conditions as the Brocard circle.

If O_1 and O_2 are points in a triangle ABC, such that

$$B\hat{C}O_2 = C\hat{A}O_2 = A\hat{B}O_2$$

and

$$C\hat{B}O_1 = A\hat{C}O_1 = B\hat{A}O_1,$$

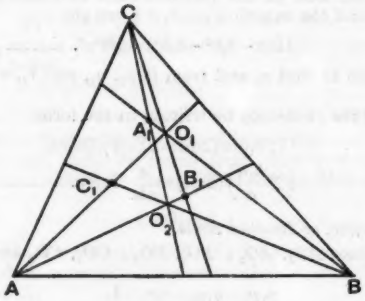
they are called the Brocard points, and if BO_1, CO_2 meet in A_1, CO_1, AO_2 in B_1, and AO_1, BO_2 in C_1, $O_1O_2A_1B_1C_1$ are concyclic, and the circle on which they lie is the Brocard circle.

Defining A_1, B_1, C_1 in the same way, can we find any other positions of O_1 and O_2 which make the five points O_1, O_2, A_1, B_1, C_1 concyclic? To answer this question we express the equation to the conic through these five points in trilinear coordinates, and write down the condition that it shall be a circle.

Let the equations of AO_1, BO_1, CO_1 be

$$\beta = p_1\gamma, \quad \gamma = q_1\alpha, \quad \alpha = r_1\beta,$$

and of AO_2, BO_2, CO_2 be $\beta = p_2\gamma$, $\gamma = q_2\alpha$, $\alpha = r_2\beta$.



Since the first group of lines meet in O_1,

$$p_1q_1r_1 = 1.$$

Similarly,

$$p_2q_2r_2 = 1.$$

Any conic through $O_1O_2A_1B_1$ can be written

$$(\gamma - q_1\alpha)(\beta - p_2\gamma) = k(\alpha - r_1\beta)(\alpha - r_2\beta).$$

From the above equations the coordinates of C_1 are proportional to

$$(1,\ p_1q_2,\ q_2).$$

Hence C_1 lies on the conic if

$$(q_2 - q_1)(p_1q_2 - p_2q_2) = k(1 - p_1r_1q_2)(1 - p_1q_2r_2);$$

therefore

$$k = q_1q_2p_2 = q_1/r_2.$$

The equation of the conic is therefore

$$q_1/r_2 \cdot \alpha^2 + q_1r_1\beta^2 + p_2\gamma^2 - \beta\gamma - p_2q_1\gamma\alpha + \alpha\beta/r_2p_1 = 0.$$

Multiplying throughout by $[(r_2p_1)/(p_2q_1)]^{\frac{1}{2}}$, this equation simplifies to the more symmetrical form

$$\Sigma\left(\frac{q_1q_2}{r_1r_2}\right)^{\frac{1}{2}}a^2-\Sigma\left(\frac{r_2p_1}{p_2q_1}\right)^{\frac{1}{2}}\beta\gamma=0. \quad \ldots\ldots(1)$$

(1) is considerably simplified by writing

$$p_1p_2=\frac{k_3}{k_2},\quad q_1q_2=\frac{k_1}{k_3},\quad r_1r_2=\frac{k_2}{k_1},$$

and $$k_1k_2k_3=1;$$

it then becomes $$\Sigma k_1a^2-\Sigma p_1k_2\beta\gamma=0. \quad \ldots\ldots(2)$$

(2) is a circle if

$$\begin{aligned}k_2c^2+k_3b^2+p_1k_2bc&=k_3a^2+k_1c^2+k_3q_1ca\\&=k_1b^2+k_2a^2+k_1r_1ab\\&=x, \text{ say,} \quad \ldots\ldots(3)\end{aligned}$$

The equations (3) enable us to find k_1, k_2, and k_3 corresponding to any given p_1, q_1, and r_1; in other words, to find the positions of O_2 corresponding to any given O_1 which will make $O_1O_2A_1B_1C_1$ concyclic. Thus when O_1 is the incentre $p_1=q_1=r_1=1$, and k_1, k_2 and k_3 are then proportional to

$$a(ab+c^2-a^2),\quad b(bc+a^2-b^2),\quad c(ca+b^2-c^2),$$

so that $$p_2=c(ca+b^2-c^2)/(b(bc+a^2-b^2).$$

This result is not of particular interest, nor are those obtained by taking O_1 to be the centroid or the circumcentre.

We can, however, look at the problem in a different way. Eliminating p_1, q_1, r_1 from (3) and the equation $p_1q_1r_1=1$, we get

$$\Pi(x-k_2c^2-k_3b^2)=a^2b^2c^2. \quad \ldots\ldots(4)$$

(4) is an equation to find x, and from it p_1, q_1, and r_1, when k_1, k_2, and k_3 are given.

The equation of the circle can be written in the form

$$(\Sigma k_1a/a)\Sigma aa=x\Sigma a\beta\gamma/(abc),$$

$$\left(\Sigma\frac{k_1a}{a}\right)(\Sigma aa)=\frac{x}{abc}\Sigma a\beta\gamma. \quad \ldots\ldots(5)$$

The isogonal conjugate, or Brocard circle.

If the pairs of lines AO_1, AO_2; BO_1, BO_2; CO_1, CO_2 are pairs of isogonal conjugates,

$$p_1p_2=q_1q_2=r_1r_2=1,$$

hence $$k_1=k_2=k_3=1.$$

(4) therefore becomes $$\Pi(x-b^2-c^2)=a^2b^2c^2.$$

which can be written $(x-\Sigma a^2)(x^2-x\Sigma a^2+\Sigma b^2c^2)=0.$

$x=\Sigma a^2$ makes $p_1=\dfrac{a^2}{bc}$ and $p_2=\dfrac{bc}{a^2}$.

The other values of x are imaginary, since $(\Sigma a^2)^2>\Sigma b^2c^2$ if

$$a+b>c,\quad b+c>a,\quad \text{and}\quad c+a>b.$$

Thus only one real circle can be drawn if O_1 and O_2 are isogonal conjugate points, and this circle is therefore the Brocard circle.

The isotomic conjugate circle.

(5) shows that the radical axis of the circle and the circumcircle is

$$\Sigma k_1a/a=0.$$

This is the line at infinity if

$$k_1/a^3 = k_2/b^3 = k_3/c^3 = 1/(abc)^{\frac{3}{2}},$$

and the circle is then concentric with the circumcircle.

When $$p_1 p_2 = k_2/k_3 = c^3/b^3,$$

the lines AO_1, AO_2 cut BC in points M and N such that $BM = CN$. Such lines are called isotomic conjugates. We may therefore call this circle the "isotomic conjugate circle."

With the above values of the k's (4) becomes

$$\Pi(y - 2b^2c^2) = a^4b^4c^4,$$

y being written for $x(abc)^{\frac{2}{3}}$. This equation has no simple solution in terms of a, b, and c.

Circles connected with the orthocentre and the Brocard points.

If $$k_1/(ab^2\cos A) = k_2/(bc^2\cos B) = k_3/(ca^2\cos C) = 1/(abck),$$

where $$k^3 = \cos A \cos B \cos C,$$

$$x - k_2c^2 - k_3b^2 = x - bc/(abck)\,(ac^3\cos B + A^3b\cos C)$$
$$= x - 1/(2a^2k)[a^2(a^2+b^2+c^2) + c^2(a^2+c^2-b^2)]$$
$$= x - 1(a^2+b^2+c^2)/(2k) + bc^2\cos c/(ak).$$

Hence, if $$x = \Sigma a^3/(2k),$$

the left-hand side of (4) becomes

$$\Pi bc^2\cos C/(ak) = a^2b^2c^2.$$

(4) is therefore satisfied by this value of x; using it we find

$$p_1 = \cos C/\cos B, \quad \text{and} \quad p_2 = a^2/(bc).$$

This value of p_1 makes O_1 the orthocentre and the value of p_2 makes O_2 one of the Brocard points.

In the same way, if $k_1/(ac^2\cos k) = \text{etc.}$,

$$x = \Sigma a^3/(2k), \quad \text{and} \quad p_1 = bc/a^2, \quad p_2 = \cos C/\cos B,$$

so that O_1 is the other Brocard point, and O_2 the orthocentre.

These two circles can easily be found from pure geometry, but they do not seem to have been noticed before. Thus for the first of them it is easy to see that if

$$\hat{CBO}_2 = \theta,$$

$\hat{CA_1B}$, $\hat{AB_1C}$, and $\hat{BC_1A}$ are equal to $90° \pm \theta$, the sign depending on whether the angle is internal or external. This shows that A_1 and B_1 lie on the circle through O_1, O_2, and C_1.

A circle coaxal with the nine-points circle and the circumcircle.

There is one real circle corresponding to a set of values of k_1, k_2, k_3. In other words, there is one circle satisfying the assumed conditions which has a given line as its radical axis with the circumcircle.

When $$k_1/a\cos A = k_2/b\cos B = k_3/c\cos C,$$

the circle, the circumcircle, and the nine-points circle have the same radical axis

$$\Sigma a\cos Aa = 0.$$

In this case the equation for x has a solution which is a simple function of a, b, and c, because it reduces to

$$\{x - (abc)^{\frac{2}{3}}/k\}^3 = a^2b^2c^2,$$

making $$x = (abc)^{\frac{2}{3}}(1 + 1/k).$$

This makes $p_1 = ck/(b\cos c)$, and $p_2 = \cos^2 c/(k\cos B)$.

H. V. LOWRY.

A SIMPLE THEORY OF THE GAMMA-FUNCTION.

By Albert Eagle, B.Sc.

1. *Introduction.* Many students interest themselves in finding a function which is equal to factorial n for integral values of n. There is a certain fascination about this function: it is so different from the transcendental functions of applied mathematics.

The following is an attempt to popularise the function by giving a simple treatment that can be put before a student quite early in his mathematical career. The function, not being one defined by a differential equation, permits of this being done much more readily than in the case of most other transcendental functions. It does not actually occur in mathematics till the study of definite integrals is reached, but this stage is generally considered as being too early to present its theory.

2. *Notation.* The function which is equal to $\lfloor x$ or $x!$ for positive integral values of x is known as the Gamma-function, and is denoted by $\Gamma(x+1)$. Probably most mathematicians wish that what is denoted by $\Gamma(x+1)$ had been denoted by $\Gamma(x)$; but the notation cannot now be changed. Gauss called it $\Pi(x)$, but since his days Π has become generally used for the product of a number of factors, just as Σ is used for the sum of a number of terms. For ordinary writing and teaching purposes nothing more convenient than $\lfloor x$ is required; but the typographical objections to this otherwise exceedingly handy notation cause us to revert to the Gaussian form. Moreover, just as $\Sigma(x)$ without any indication of limits is naturally interpreted as being $1+2+3+\ldots+x$, when x is an integer, so in that case may $\Pi(x)$, without any indication of limits, be naturally interpreted as meaning $1\,.\,2\,.\,3\,.\ldots\,x$, and to represent the required function in other cases. Lastly, as far as the name of the function is concerned, the term "factorial-function" is certainly far more expressive and appropriate than the merely alphabetical designation of "gamma-function" for one of the most notable functions in mathematics.

3. *Utility of the Function.* Three chief purposes for which the factorial function is useful in mathematics:

(1) It enables us to express the product of any number of factors in A.P.—which occurs so frequently in the coefficients of series—in a simple form. Thus

$$(a+b)(a+2b)\ldots(a+nb)=b^n\Pi(a/b+n)\div\Pi(a/b).*$$

(2) It enables us to interpolate the preceding result for a non-integral number of factors, just as $\exp(kn)$ enables us to interpolate $(e^k)^n$ for any non-integral value of n.

(3) It enables us to express many important definite integrals in terms of the factorial function.

4. *Definition of* $\Pi(x)$. To begin with we will make the function $\Pi(x)$ satisfy the two equations

$$\Pi(0)=1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$\Pi(x)=x\Pi(x-1), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

the latter equation being considered as holding for *all* values of x, integral, or real, or not. When x is integral the combination of (1) and (2) gives

$$\Pi(x)=1\,.\,2\,.\,3\ldots x\equiv x!$$

The equations (1) and (2) do not define a definite function at all, for we might make it vary in any manner we chose over any given unit range and then use (2) to extend the function to $-\infty$ and $+\infty$.

* The reciprocal of this equation gives the product of a number of factors in H.P., while the product of n factors in G.P. is very easily expressible in terms of $\exp(n^2)$ and $\exp(n)$.

The question is, Which of all these functions will occur most frequently or be of most importance in mathematics? We can easily prove that the ratio, $R(x)$, of any two functions $G_1(x)$ and $G_2(x)$ satisfying (1) and (2) is a *periodic function* (that is an *oscillating function*) whose period is unity or a sub-multiple thereof. For we have

$$R(x+1)=\frac{G_1(x+1)}{G_2(x+1)}=\frac{(x+1)G_1(x)}{(x+1)G_2(x)}=\frac{G_1(x)}{G_2(x)}=R(x);$$

whence $R(x)$, if it is not a mere constant, is a periodic function whose period is unity or an aliquot part thereof. Hence if we know any one function satisfying (2) we can obtain all others by multiplying it by some periodic function.

Now, if x is a positive integer

$$\frac{\Pi(n+x)}{\Pi(n)}\equiv\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{x}{n}\right).$$

If then we make $n\to\infty$ the right-hand side clearly $\to 1$, x being finite. Hence all functions satisfying (1) and (2) satisfy the equation

$$\lim_{n\to\infty}\frac{f(n+x)}{n^x f(n)}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

for all positive integral values of x. Now this equation is intelligible for non-integral values of x, and gives a unique relation between $f(n+x)$ and $f(n)$ in that case also if n^x is taken exclusively at its principal value of $\exp(x\log n)$. Hence (3) taken in this manner in conjunction with (1) and (2) defines a unique single-valued function from $-\infty$ to $+\infty$.

This function we proceed to study; and it will appear later that this is the oft-arising function for which we are in search.

5. *Approximate Calculation of* $\Pi(x)$. Let us plot, for integral values of x, the points given by $y=\log\Pi(x)\equiv\log 1+\log 2+\ldots+\log x$. The "smooth curve" through these points representing $y=\log\Pi(x)$ will evidently be concave upwards for all positive values of x. Let P, Q and R denote the points for $x=n-1$, n and $n+1$ respectively, so that the slope of PQ is $\log n$ and of QR is $\log(n+1)$; also let $0<f<1$.

Then since, between Q and R, the curve is above the secant PQ and below the chord QR, we have

$$\log\Pi(n)+f\log n<\log\Pi(n+f)<\log\Pi(n)+f\log(n+1),$$

i.e.

$$n^f\Pi(n)<\Pi(n+f)<(n+1)^f,$$

whence $\Pi(n+f)$ calculated from

$$\Pi(n+f)\doteqdot n^f\Pi(n)$$

is in error by less than f parts in n; and hence any $\Pi(x)$ calculated from $\Pi(n+f)$ by repeated applications of (2) is in error by less than f parts in n parts, which can be made as small as we like by choosing n sufficiently large.

6. *Some General Theorems.* Before proceeding to obtain computable expressions for $\Pi(x)$, which we henceforth take as the function satisfying the three equations:

$$f(0)=1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4\text{ A})$$

$$f(x-1)=f(x)/x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4\text{ B})$$

and

$$\lim_{n\to\infty}\frac{f(n+x)}{n^x f(n)}=1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4\text{ C})$$

we will give some general theorems on functions satisfying one or more of these relations.

THEOREM I. If $f(x)$ satisfies (4 C) and $F(x)$ satisfies the condition

$$\lim_{x\to\infty}\frac{F(x)}{f(x)}=1,$$

then $F(x)$ satisfies (4 C). We regard this as obvious.

THEOREM II. If $f(x)$ satisfies (4 C), then $f(x+a)$, where a is any constant, satisfies (4 C).

By hypothesis

$$\lim_{m\to\infty}\frac{f(m+x)}{m^x f(x)}=1;$$

writing $m=n+a$ we get

$$\lim_{n\to\infty}\frac{f(n+x+a)}{(n+a)^x f(n+a)}=1.$$

Since

$$\lim_{n\to\infty}\left(\frac{n}{n+a}\right)^x=1,$$

we get

$$\lim_{n\to\infty}\frac{f(n+x+a)}{n^x f(n+a)}=1,$$

proving the proposition.

THEOREM III. If $f(x)$ satisfies (4 C) then $cx^q f(x)$, where c and q are constants, satisfies (4 C).

For

$$\lim_{n\to\infty}\frac{c(n+x)^q f(n+x)}{n^x c n^q f(n)}=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^q\cdot\frac{f(n+x)}{n^x f(n)}=1,$$

since the limit of the first factor is unity, and by hypothesis $f(x)$ satisfies (4 C).

THEOREM IV. If $f_1(x)$ and $f_2(x)$ are any two functions satisfying (4 C), then $2^x f_1(\frac{1}{2}x)\,.\,f_2(\frac{1}{2}x)$ satisfies (4 C).

For this function will satisfy (4 C) provided

$$\lim_{m\to\infty}\frac{2^{m+y}f_1\left(\frac{m+y}{2}\right)f_2\left(\frac{m+y}{2}\right)}{m^y 2^m f_1\left(\frac{m}{2}\right)f_2\left(\frac{m}{2}\right)}=1.$$

Writing $m/2\equiv n$ and $y/2\equiv x$, this becomes

$$\lim_{n\to\infty}\frac{f_1(n+x)f_2(n+x)}{n^{2x}f_1(n)f_2(n)}=1;$$

which is evidently true since both f_1 and f_2 satisfy (4 C).

COR. An exactly similar proof will show that, more generally, if $f_1, f_2 \cdots f_p$ are p functions satisfying (4 C), the expression

$$p^x f_1(x/p)\,.\,f_2(x/p)\cdots f_p(x/p)$$

will likewise satisfy it.

THEOREM V. We have $k_2 2^x \Pi(\frac{1}{2}x)\Pi(\frac{1}{2}x-\frac{1}{2})=\Pi(x)$, where $k_2^{-1}=\Pi(-\frac{1}{2})$. Denoting the left-hand side by $f(x)$, evidently $f(0)=1$. Changing x into $x-1$ we get

$$\frac{f(x)}{f(x-1)}=\frac{2\Pi(\frac{1}{2}x)\Pi(\frac{1}{2}x-\frac{1}{2})}{\Pi(\frac{1}{2}x-\frac{1}{2})\Pi(\frac{1}{2}x-1)}=2\,.\,\tfrac{1}{2}x.$$

Hence $f(x)$ satisfies both (4 A) and (4 B); by Theorems II. and IV. it satisfies (4 C) also, hence $f(x)=\Pi(x)$.

Cor. 1. In a similar manner we may prove Gauss's theorem that

$$k_p p^x \Pi\left(\frac{x}{p}\right)\Pi\left(\frac{x-1}{p}\right)\ldots\Pi\left(\frac{x-p+1}{p}\right)=\Pi(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $$k_p^{-1}\equiv\Pi\left(-\frac{1}{p}\right)\Pi\left(-\frac{2}{p}\right)\ldots\Pi\left(-\frac{p-1}{p}\right).$$

For the left-hand side evidently satisfies (4 A), and (by Theorem IV. Cor.), (4 C); changing x into $x-1$ gives us

$$\frac{f(x)}{f(x-1)}=\frac{p\Pi(x/p)}{\Pi(x/p-1)}\equiv x;$$

whence (4 B) is satisfied; and hence the result. It will be shown later (p. 126) that

$$k_p^{-1}=(2\pi)^{\frac{p-1}{2}}/p^{\frac{1}{2}}.$$

Cor. 2. Using the above value for k_p, we have, when x is large,

$$p^x\{\Pi(x/p)\}^p\sim\frac{(2\pi x/p)^{\frac{p-1}{2}}\Pi(x)}{\sqrt{p}}\ *. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

For when x is large we have

$$\Pi\left(\frac{x-m}{p}\right)\sim\left(\frac{x}{p}\right)^{-\frac{m}{p}}\Pi(x/p);$$

transforming all the terms on the left-hand side of (5) by this relation, we get

$$k_p p^x\{\Pi(x/p)\}^p=(x/p)^{\frac{p-1}{2}}\Pi(x);$$

whence, substituting the above value of k_p, the result follows.

Equation (6) when $p=2$ or $\frac{1}{2}$, or 3 or $\frac{1}{3}$, gives Wallis's value for π when $x\to\infty$, although this is more easily obtainable by simply putting $x=\frac{1}{2}$ in (18) below.

Theorem VI. The function $(x/e)^x$ satisfies the condition (4 C). For this gives

$$\frac{f(n+x)}{n^x f(x)}=n^{-x}\left(\frac{n+x}{e}\right)^{n+x}\left(\frac{e}{x}\right)^n\equiv\left(1+\frac{x}{n}\right)^x\left(1+\frac{x}{n}\right)^n e^{-x}.$$

In the limit, as $n\to\infty$, this becomes $1\,.\,e^x\,.\,e^{-x}\equiv 1$; hence the result.

Cor. 1. By Theorem II. the function $cx^q(x/e)^x$ satisfies (4 C).

Cor. 2. If *any* function $f(x)$ satisfies both (4 A) and (4 B) *and the condition*

$$\lim_{x\to\infty}\frac{f(x)}{cx^q(x/e)^x}=1\ \dagger$$

for any values of c and q, it is the factorial function.

This follows immediately by Theorem I.

Cor. 3. The conditions of Cor. 2 can be satisfied for only one particular pair of values of c and q, if at all.

Cor. 4. Assuming the value of k_p in Theorem V. Cor. 1, the values of c and q are $\sqrt{2\pi}$ and $\frac{1}{2}$ respectively.

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* The symbol $\sim$ means "approximates to a ratio of equality with," and must not be confused with $\doteq$ or $\approx$, which mean "approximately equal to." When followed by a series in ascending or descending powers of x, the words "when x is sufficiently small or large" (respectively) are implied; in other cases the necessary qualifying condition is stated. Unlike the other two symbols, $\sim$ implies that any assignable degree of accuracy can be obtained by taking x *sufficiently* large or small as the case may be.

† It is the essence of this condition that x is not restricted to being an integer.

For replacing the $\Pi(x)$ in (6) by $cx^q(x/e)^x$, and similarly the $\Pi(x/p)$, it becomes

$$p^x c^p \left(\frac{x}{p}\right)^{pq} \left(\frac{x}{pe}\right)^x \sim \frac{(2\pi x)^{\frac{p-1}{2}}}{p^{p/2}} cx^q \left(\frac{x}{e}\right)^x;$$

which is possible, provided

$$\frac{c^{p-1}x^{pq}}{p^{pq}} = \frac{(2\pi)^{\frac{p-1}{2}} x^{\frac{p-1}{2}+q}}{p^{p/2}};$$

that is, provided $c=\sqrt{2\pi}$ and $q=\frac{1}{2}$.

Cor. 5. This *proves* that $\Pi(x) \sim \sqrt{2\pi x}(x/e)^x$ when x is large, and that no other function, such as some power of $\log x$, needs to be introduced into $cx^q(x/e)^x$ to make it tend to equality with $\Pi(x)$. This very important relation is known as Stirling's Theorem, and is generally proved by approximately evaluating, when x is large, some definite integral which is expressible in terms of $\Pi(x)$.

7. *Computable Product Formulæ for* $\Pi(x)$. Equation (3) is not, unfortunately, of much practical value as it stands, since to get a six-figure accuracy in $\Pi(n+x)$ we have seen in § 5 that we must have n of the order of a million, which would require about a million applications of the equation (2) to calculate the function for a moderate value of the argument. However, inverting (3) and multiplying both sides by $\Pi(x)$, we get

$$\begin{aligned}\Pi(x) &= \mathop{L}_{n\to\infty} \frac{n^x \Pi(n)\Pi(x)}{\Pi(n+x)} \\ &= \mathop{L}_{n\to\infty} \frac{1.2.3.\ldots.n}{(x+1)(x+2)\ldots(x+n)} \cdot n^x \\ &= \mathop{L}_{n\to\infty} \frac{n^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)}, \quad \ldots\ldots\ldots\ldots\ldots(7)\end{aligned}$$

a result due to Euler, and which the above analysis shows to be a natural definition of $\Pi(x)$.

If in (7) we replace n^x by

$$\left\{\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\right\}^x;$$

and remember that since

$$\mathop{L}_{n\to\infty}\left(\frac{n+1}{n}\right)^x = 1,$$

this may be introduced as an additional factor, we get

$$\Pi(x) = \mathop{L}_{n\to\infty} \frac{\left(1+\frac{1}{1}\right)^x\left(1+\frac{1}{2}\right)^x\ldots\left(1+\frac{1}{n}\right)^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)}; \quad \ldots\ldots\ldots\ldots\ldots(8)$$

another result due to Euler, and one which is an absolutely convergent product for all real or complex values of x, save negative integers, if each factor in the numerator is associated with the one below it in the denominator.*

* When n is large, the factors approximate to equality with $1+x(x-1)/2n^2$, and therefore their logarithms, to $x(x-1)/2n^2$; so the sum of the logarithms is finite inasmuch as the series Σn^{-2} is convergent.

Finally, another definition of $\Pi(x)$ may be obtained if we write n^x in (7) in the form $e^{x \log n}$ and then make use of the result that

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)\to\gamma,$$

where $\gamma\equiv 0{\cdot}5772157\ldots$ and is known as Euler's constant.* If we then associate the factor $e^{x/q}$ with the factor $(1+x/q)$ in the denominator of (7), we get, on inverting for convenience,

$$\frac{1}{\Pi(x)}=e^{\gamma x}\left\{\left(1+\frac{x}{1}\right)e^{-x/1}\right\}\left\{\left(1+\frac{x}{2}\right)e^{-x/2}\right\}\left\{\left(1+\frac{x}{3}\right)e^{-x/3}\right\}\ldots \text{ to } \infty\ldots\ldots(9)$$

Equation (9) is known as Weierstrass's definition of the Gamma-function; it is absolutely convergent for *all* finite values of x.

8. *A Series for calculating log* $\Pi(x)$. If we take logarithms of both sides of (8) and expand all the logarithms on the right-hand side, except those of the first term in both the numerator and denominator, we get

$$\log\Pi(x)=x\log 2-\log(1+x)+\sum_2^\infty\left(\frac{x-x}{n}-\frac{x-x^2}{2n^2}+\frac{x-x^3}{3n^3}-\ldots\right).$$

Or, dividing the last term by the factor $x(1-x)$, and using the notation of the foot-note on p. 123, we have

$$\log\Pi(x+1)=x\log 2-x(1-x)\left\{\left(\frac{S_2-1}{2}-\frac{S_3-1}{3}+\ldots\right)\right.$$
$$\left.+x\left(\frac{-S_3-1}{3}+\frac{S_4-1}{4}\ldots\right)+x^2\left(\frac{S_4-1}{4}-\frac{S_5-1}{5}+\ldots\right)+\ldots\right\}.$$

By the value for γ in the same foot-note, the coefficient of x^0 in the brackets ( ) is $\gamma-(1-\log 2)$; the coefficient of x^1 is this quantity minus $\frac{S_2-1}{2}$, and so on. Whence, putting in numerical values for each S from J. B. Dale's *Tables of Mathematical Functions*, and reducing the logarithms to common ones, we get

$$\log_{10}\Pi(1+x)=0{\cdot}3010300x-x(1-x)\{0{\cdot}1174171$$
$$-\ 0{\cdot}0226286x\ +0{\cdot}0066222x^2$$
$$-\ 0{\cdot}0023160x^3+0{\cdot}0008915x^4$$
$$-\ 0{\cdot}0003638x^5+0{\cdot}0001542x^6$$
$$-\ 0{\cdot}0000672x^7+\ldots\quad\}.\ \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

This formula is convergent for $-2<x<2$, but it need never be used except when $-\frac{1}{2}<x<\frac{1}{2}$, in which region the above terms will give a seven-figure accuracy in the logarithm.

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* It is obvious geometrically from the rectangular hyperbola that

$$1+\frac{1}{2}+\ldots+\frac{1}{n}>\int_0^n\frac{dx}{x+1}>\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

The difference between the first two terms of this inequality is clearly equal to

$$\int_0^1\left\{\left(1-\frac{1}{1+x}\right)+\left(\frac{1}{2}-\frac{1}{2+x}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+x}\right)\right\}dx.$$

Expanding $\frac{1}{p+x}$ in the form $\frac{1}{p}-\frac{x}{p^2}+\frac{x^2}{p^3}-\ldots$ and integrating term by term, this integral is found to be

$$\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)-\frac{1}{3}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)+\frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)-\ldots.$$

But the integral, in the limit, is equal to γ, which is thus seen to be equal to

$$\frac{1}{2}S_2-\frac{1}{3}S_3+\frac{1}{4}S_4-\ldots$$

where $\qquad S_p=1+2^{-p}+3^{-p}+\ldots.$

9. *Proof that* $\Pi(x)$ *is an Analytic Function.* We will now show that the function $\Pi(x)$ is an *analytic function* in that it possesses a derivative.*

If we differentiate (4 C) logarithmically we get

$$\lim_{n\to\infty}\left\{\frac{\Pi'(n+x)}{\Pi(n+x)}-\log n\right\}=0.$$

Since the second term is independent of x, the first term must also be so in the limit. Hence, taking x as zero we have

$$\frac{\Pi'(n)}{\Pi(n)}\sim\log n \quad \text{.....(11)}$$

when n is large.

If we similarly differentiate (4 B) we get

$$\frac{\Pi'(x)}{\Pi(x)}=\frac{1}{x}+\frac{\Pi'(x-1)}{\Pi(x-1)}. \quad \text{.....(12)}$$

Repeated application of this formula gives us

$$\frac{\Pi'(n)}{\Pi(n)}=\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{2}+\frac{1}{1}+\frac{\Pi'(0)}{\Pi(0)}, \quad \text{.....(13)}$$

when n is an integer.

Subtracting (11) from (13) and making $n\to\infty$ we get

$$\frac{\Pi'(0)}{\Pi(0)}=-\gamma;$$

and hence, when x is an integer,

$$\frac{\Pi'(x)}{\Pi(x)}=1+\frac{1}{2}+\ldots+\frac{1}{x}-\gamma. \quad \text{.....(14)}$$

If x is not an integer, repeated application of (12) will still give

$$\frac{\Pi'(n+x)}{\Pi(n+x)}=\frac{1}{n+x}+\frac{1}{n-1+x}+\ldots+\frac{1}{x+1}+\frac{\Pi'(x)}{\Pi(x)}; \quad \text{.....(15)}$$

but we have just seen that the left-hand side, in the limit, is independent of x and equal to $\Pi'(n)/\Pi(n)$; whence substituting the value of this from (14) in (15), we get

$$\frac{\Pi'(x)}{\Pi(x)}=-\gamma+\left(1-\frac{1}{x+1}\right)+\left(\frac{1}{2}-\frac{1}{x+2}\right)+\left(\frac{1}{3}-\frac{1}{x+3}\right)+\ldots.\dagger \quad \text{.....(16)}$$

Keeping the two terms in the brackets associated, this series is seen to be absolutely convergent for all values of $x>-1$. Noting that the left-hand side of (16) is $\frac{d}{dx}\log\Pi(x)$ and differentiating again, we get

$$\frac{d^2}{dx^2}\log\Pi(x)=\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\ldots, \quad \text{.....(17)}$$

which is also an absolutely convergent series for all values of $x>-1$.

* If a function is considered only for real values of the variable, to prove that it is analytic in any interval, it is necessary to prove that it possesses derivatives *of all orders* at all points within the interval; if we are considering the function for complex values of the variable it is only necessary, in order to prove that it is analytic over any region, to prove that it possesses a *first derivative* at all points of that region, when it can be shown to possess derivatives of all orders at all points *within* that region.

† It should be noted that (16) is the interpolation formula that the Calculus of Finite Differences would give for a function satisfying (14). For if, for integral values of x,

$$f(x)\equiv u_1+u_2+\ldots+u_x,$$

we have

$$f(x)=(E+E^2+\ldots+E^x)\,.\,u_0\equiv(E-E^{x+1})(1-E)^{-1}\,.\,u_0\equiv(E-E^{x+1})(1+E+E^2+\ldots)\,.\,u_0$$
$$\equiv(u_1-u_{x+1})+(u_2-u_{x+2})+\ldots.$$

The expressions for the higher derivatives of $\log \Pi(x)$ are obvious; and the values of the derivatives at $x=0$ will give us, by Maclaurin's Theorem,

$$\log \Pi(x) = -\gamma x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \dots; \quad \dots\dots\dots\dots(18)$$

of which (10) is merely a convenient modification for practical calculation.

Finally, if we put $x=-\frac{1}{2}$ in (16), it becomes

$$\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})} = -\gamma + 2\{(\tfrac{1}{2}-\tfrac{1}{1}) + (\tfrac{1}{4}-\tfrac{1}{3}) + (\tfrac{1}{6}-\tfrac{1}{5}) + \dots\};$$

that is $$\Pi'(-\tfrac{1}{2})/\Pi(-\tfrac{1}{2}) = -\gamma - 2\log 2 = -1{\cdot}96351003\dots;$$

and then by (12), $$\Pi'(\tfrac{1}{2})/\Pi(\tfrac{1}{2}) = +0{\cdot}0364900\dots.$$

The comparative smallness of the latter value compared with the former is due to the fact that $\log \Pi(x)$ (and therefore $\Pi(x)$) passes through a minimum value of $\Pi(x)=0{\cdot}88560\dots$ when $x=0{\cdot}46163\dots$. It is shown later (p. 126) that $\Pi(-\frac{1}{2})=\sqrt{\pi}$ and $\Pi(\frac{1}{2})=\frac{1}{2}\sqrt{\pi}$.

We see from (16 and (17) that all orders of derivatives of $\log \Pi(x)$ exist throughout the interval from -1 to $+\infty$; and hence $\log \Pi(x)$ (and therefore $\Pi(x)$) is an analytic function of x throughout that interval. Considering complex values, we see that (16) converges for all such values except negative integers; and hence $\Pi(x)$ is a single-valued analytic function at all finite points of the complex plane, excluding those corresponding to negative integers.

We will remark in passing that the equation (16) may be deduced with less formality by differentiating (9) logarithmically; and that if we similarly differentiate (7) we get

$$\frac{\Pi'(x)}{\Pi(x)} = \mathop{L}_{n\to\infty}\left\{\log n - \left(\frac{1}{1+x}+\frac{1}{2+x}+\dots+\frac{1}{n+x}\right)\right\}.$$

A first approximation to the terms in brackets, when x is large, is obviously $\log\{(n+x)/x\}$; but it is evident, geometrically, that a much better approximation is $\log\{(n+x+\frac{1}{2})/(x+\frac{1}{2})\}$. Substituting this value, we have

$$\frac{\Pi'(x)}{\Pi(x)} \doteq \log(x+\tfrac{1}{2}) \doteq \log x + \frac{1}{2x},$$

when x is large. Treating the $\doteq$ as $=$ and solving for $\Pi(x)$, we get

$$\Pi(x) = Cx^{\frac{1}{2}}(x/e)^x,$$

where C is some constant, which agrees with the form given by Stirling's theorem.

10. *Connection of* $\Pi(x)$ *with the Trigonometric Functions.* The product of the two functions $\Pi(x)$ and $\Pi(-x)$ is intimately connected with the sine function. From the infinite product formula,

$$\sin \pi x = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\dots\dots\dots\dots(19)$$

we get immediately by (9),

$$\frac{1}{\Pi(x)\Pi(-x)} = \frac{\sin \pi x}{\pi x}; \quad \dots\dots\dots\dots(20)$$

which may be written

$$\frac{1}{\Pi(x-1)\Pi(-x)} = \frac{\sin \pi x}{\pi}; \quad \dots\dots\dots\dots(20\text{ A})$$

or again, by writing $x=z+\frac{1}{2}$,

$$\Pi(z-\tfrac{1}{2})\Pi(-z-\tfrac{1}{2}) = \pi \sec \pi z. \quad \dots\dots\dots\dots(20\text{ B})$$

Hence a factorial for an argument $< -\frac{1}{2}$ is immediately expressible in terms of one of argument $> -\frac{1}{2}$: it should also be noted that the product of *any two* factorials the *sum* of whose arguments is an integer is readily obtainable in a trigonometric form; thus

$$\Pi(1\cdot3)\,.\,\Pi(1\cdot7)=1\cdot3\,.\,0\cdot3\,.\,0\cdot7\,.\,1\cdot7\,.\,\pi\sec\pi/5=1\cdot8022\ldots.$$

Putting $z=0$ in (20 B) we get at once $\Pi(-\frac{1}{2})=\sqrt{\pi}=1\cdot772454\ldots$, and therefore $\Pi(\frac{1}{2})=\frac{1}{2}\sqrt{\pi}=0\cdot8862272\ldots$.

The behaviour of $\Pi(x)$ for negative values of x may now very easily be seen by writing (20) in the form

$$\Pi(-x)=\frac{\pi}{\Pi(x-1)}\operatorname{cosec}\pi x\,;$$

or we may say in words, that the $\Pi(-x)$ curve is the reciprocal of the $\Pi(x-1)\,.\,\dfrac{\sin\pi x}{\pi}$ curve.

We are now in a position to evaluate the constant k_p^{-1} (p. 121), which was equal to

$$\Pi(-1/p)\Pi(-2/p)\ldots\Pi(-1+1/p). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

Multiplying this expression by itself backwards, we see, by (20 A), that

$$k_p^{\,2}=\frac{\sin\pi/p}{\pi}\,.\,\frac{\sin2\pi/p}{\pi}\ldots\frac{\sin(p-1)\pi/p}{\pi}\,.$$

By a well-known result in trigonometry the numerator is equal to $p/2^{p-1}$.*

Hence $$k_p^{-1}=(2\pi)^{\frac{p-1}{2}}/\sqrt{p},$$

so that $\Pi(-\frac{1}{2})=\sqrt{\pi}$, $\Pi(-\frac{1}{3})\Pi(-\frac{2}{3})=2\pi/\sqrt{3}$, and so on. The evaluation of the expression (21) is also due to Euler.

11. *The Eulerian Integral* $\int_0^\infty e^{-t}t^x\,dt$. We will suppose that x is real and >-1, so that the integral is convergent. Denoting it by $I(x)$ we have obviously $I(0)=1$, while a simple integration by parts gives instantly

$$I(x)=xI(x-1)\,;$$

hence $I(x)$ satisfies the conditions (4 A) and (4 B). To investigate its behaviour when x is large we note that the integrand attains its maximum when $t=x$: we therefore write $t=x+u$ when

$$I(x)=\int_{-x}^{\infty}e^{-(x+u)}(x+u)^x du=x^xe^{-x}\int_{-x}^{\infty}e^{-u}\Big(1+\frac{u}{x}\Big)^x du.$$

Writing $(1+u/x)^x$ in the form $\exp\{x\log(1+u/x)\}$ and expanding the logarithm in powers of u/x, we get

$$I(x)=x^xe^{-x}\int_{-x}^{\infty}\exp\Big(-\frac{u^2}{2x}+\frac{u^3}{3x^2}-\frac{u^4}{4x^3}+\ldots\Big)du.$$

Writing further $u=v\sqrt{2x}$, we get

$$I(x)=x^xe^{-x}\sqrt{2x}\int_{-\sqrt{x/2}}^{\infty}\exp\Big(-v^2+\frac{v^3\sqrt{8}}{3\sqrt{x}}-\ldots\Big)dv. \qquad \ldots\ldots\ldots\ldots(22)$$

Making $x\to\infty$ we get

$$\lim_{x\to\infty}\frac{I(x)}{x^xe^{-x}\sqrt{2x}}=\int_{-\infty}^{\infty}e^{-v^2}dv.\dagger \qquad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

Since $I(x)$ thus tends to a ratio of equality with a function of the form

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* Obtainable, *e.g.* by dividing the equation

$$\sin p\theta=2^{p-1}\sin\theta\,.\,\sin(\theta+\pi/p)\,.\,\sin(\theta+2\pi/p)\ldots\sin(\theta+\overline{p-1}\pi/p)$$

by $\sin\theta$ and making $\theta\to0$.

† This treatment of the integral in (23) may be rigidly justified as follows: The series for the index in (22) is convergent if $v^3<x/2$. Let us consider the integral taken between the

$cx^q(x/e)^x$ when x is large (whether x is an integer or not), and since it also satisfies the conditions (4 A) and (4 B), it is, by Theorem VI. Cor. 2, the factorial function $\Pi(x)$. It only remains to evaluate the integral in (23). Writing $v^2=s$, it becomes

$$2\times\tfrac{1}{2}\int_0^\infty e^{-s}s^{-1/2}ds\,;$$

that is, $I(-\frac{1}{2})$, or $\Pi(-\frac{1}{2})$, which we have already found is $\sqrt{\pi}$. Hence we have a second proof of Stirling's Theorem that

$$\Pi(x) \sim \sqrt{2\pi x}(x/e)^x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

when x is large.

If we expand the exponential in (22) in powers of v, we can easily get a second approximation to (24). We have

$$\exp\left(\frac{v^3\sqrt{8}}{3\sqrt{x}}-\frac{v^4}{4x}\right)=1+\frac{v^3\sqrt{8}}{3\sqrt{x}}-\frac{v^4}{x}+\frac{1}{2}\left(\frac{v^3\sqrt{8}}{3\sqrt{x}}\right)^2+\ldots$$

approx. Hence (21) becomes, in the limit,

$$\int_{-\infty}^{\infty}e^{-v^2}\left(1+\frac{v^3\sqrt{8}}{3\sqrt{x}}-\frac{v^4}{x}+\frac{4}{9}\frac{v^6}{x}\right)dv.$$

The second term, since it makes an odd function of v, vanishes, while the remaining two terms give, if we write $v^2=t$,

$$\frac{2}{x}\cdot\frac{1}{2}\int_0^\infty e^{-t}\left(\frac{4}{9}t^{5/2}-t^{3/2}\right)dt\,;$$

that is,
$$\frac{1}{x}\left\{\frac{4}{9}\Pi(2{\cdot}5)-\Pi(1{\cdot}5)\right\}=\left(\frac{4}{9}\cdot\frac{5}{2}\cdot\frac{3}{2}-\frac{3}{2}\right)\frac{\sqrt{\pi}}{2x}=\frac{\sqrt{\pi}}{12x}.$$

Hence
$$\Pi(x)\sim\sqrt{2\pi x}(x/e)^x\left(1+\frac{1}{12x}\right)$$

when x is large.

The University,
Manchester.

ALBERT EAGLE.

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530. George Spencer, father of Herbert, looked forward hopefully to the time when all words, ideas, and sentiments would be stripped of their drossy and fallacious accretions, coined into a universal currency, and made amenable to mathematical calculation. He used to insist on the propriety, indeed the honesty, of always employing the same word for the same thing, and not attempting to please the ear of the hearer or the reader by the use of words not really synonymous as meaning the same. . . .—Mozley, *Reminiscences*, i. 147.

531. Roderick Random at Glasgow University learned Greek very well, and was pretty far advanced in the mathematics.

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limits $-N$ and $+N$ where $N^{10}=x$; these limits obviously $\to\infty$ with x, and within this region the index series differs from $-v^3$ by less than

$$\frac{\sqrt{8}}{3}\frac{v^3}{\sqrt{x}}\left(1-\frac{v\sqrt{2}}{\sqrt{x}}\right)^{-1}.$$

As $|v|\leqslant N$, *i.e.* $\leqslant x^{0\cdot1}$, this is less than $\frac{1}{3}\sqrt{8}x^{-0\cdot3}(1-\sqrt{2}x^{-0\cdot4})^{-1}$. But e raised to this power, when x is large, differs from unity by $<Ax^{-0\cdot3}$ where A is a finite constant tending to $\frac{1}{3}\sqrt{8}$. As this correction is applied to a factor, $\exp(-v^3)$, which never exceeds unity, the total error in the integral due to omitting all terms save the first is, in the limit, necessarily less than

$$\int_{-N}^{N}Ax^{-0\cdot3}dv\,;$$

i.e. $<2NAx^{-0\cdot3}$ or $2Ax^{-0\cdot1}$ which $\to0$ as $x\to\infty$.

GRAMMAR SCHOOL ARITHMETIC A CENTURY AGO.

THE work of writers like Rouse Ball in England, and Cajori and D. E. Smith in America, in revealing the fascinating story of the history of arithmetic, has shown the subject developing slowly but surely along the corridor of time. This development, which culminates in present-day arithmetic, proceeds not only by increase of subject-matter, but involves suppression and modification both of matter and methods. It is not the haphazard growth of the wild plant, but the steady and systematic development which results from the pruning and shaping of the skilled and sympathetic gardener.

Something of the nature of this evolution, for evolution it is, can probably best be realised by a study of the subject just as it is emerging into its present form, for our consciousness of the present is most vivid and most familiar, and the contrast between the immediate past and the present is therefore calculated to throw up strongly those changes which have turned the former into the latter. And for this purpose reality is most likely to be secured if we can transport ourselves to the historic classroom and see the subject as there presented to the pupil. It is in this hope that an account is here given of some mathematical exercise books compiled by a pupil at a village Grammar School in Yorkshire during the years 1817 to 1822. In those days the school text-book was a much more expensive and rarer article than now, and the exercise book became, in effect, the text-book, so that this pupil's books may be considered as giving a fair indication of the kind of mathematics taught in a Grammar School to boys of his age in the early part of last century. It will be noticed that the books do not cover the whole of the school period, so that the full mathematical curriculum is probably not represented. The arithmetic, however, seems complete, and can be compared and contrasted with twentieth-century arithmetic. There are five books, one for each year, containing 180, 182, 156, 68, 68 pages of rough paper. The first two books are $6\frac{1}{2}$ ins. $\times 8$ ins. in size, bound in stiff backing with leather over the hinge and corners, and with mottled paper on the outside face of each back. The mottled paper has Bible letterpress printed on it. Price of each book 2s. 6d. The remaining books are $7\frac{1}{2}$ ins. $\times 9$ ins., and the two latest have flexible backs.

In the case of all the books there are no printed lines, but the pupil has drawn thin pencil lines for all his writing and figuring, and has used red ink freely in the manner which was common until fairly recently, viz.: Single lines for separating elements of working, double or multiple lines for indicating an answer. This lavish use of lines, along with the large florid copperplate headlines, the exceedingly legible small-hand writing, the artistically formed figures, the very careful spacing, arrangement and numbering of items, and the detail with which the subject-matter is developed, reflects the permanent character of the record and the comparative leisureliness of the period.

The contents of each book consist of a series of topics, each of which is introduced by definition, then explained by statement of "Rule," and, finally, illustrated with a number of worked examples. The "Rule," which nowadays has been dethroned from its absolute position, here reigns supreme, and ensures that the pupil shall never be left to the guidance and stimulus of his own initiative. Even in the case of an individual topic the illustrated examples are graded most carefully and, in many cases, classified under various rules. Instances will be given later in considering the books singly. One cannot read some of these definitions and rules without feeling much sympathy for the pupil and rejoicing that we do nowadays realise the necessity for giving a pupil practical experience of circumstances defined and reduced to rule before we present the definition and rule formally.

Book I. April 1817 *to August* 1817. *Age of pupil*, 11.

The subject-matter consists of the topics given below, in the order given. The brackets indicate the number of pages and the number of illustrative

examples respectively. Both Rules and Examples are carefully numbered, and the latter are totalled (wrongly) at the end of the book. There is an almost complete absence of fractions. Decimals are unmentioned, and there is no use of vulgar fractions except on two or three occasions when a remainder is expressed as a fraction of the divisor.

Inverse Proportion (10, 10), Double Rule of Three (11, 8), Practice (55, 108), Simple Interest (10, 10), Commission (4, 3), Purchasing Stock (6, 5), Brokage (4, 3), Simple Interest (23, 17), Compound Interest (15, 6), Rebate or Discount (24, 10), Equation of Payments (8, 5), Barter (12, 8), Profit and Loss (7, 5).

The definition of "Inverse Proportion" is a gem:

"Inverse proportion is when more requires less, and less requires more. More requires less, is when the third term is greater than the first, and requires the fourth term to be less than the second. And less requires more, is when the third term is less than the first, and requires the fourth term to be greater than the second."

The Rule is: "Multiply the first and second terms together, and divide the product by the third; the quotient will bear such proportion to the second as the first does to the third." There are no short methods used, all the calculations being laboriously carried out according to rule, thus:

"As 3 : 1000 :: 6

3

6) 3000

500 Answer.

The rules for "Double Rule of Three" are worth quoting:

Rule 1. Let the principal cause of loss or gain, interest or decrease, action or passion, be put in the first place.

2. Let that which betokeneth time, distance of place and the like, be in the second place, and the remaining one in the third.

3. Place the other terms under their like in the supposition.

4. If the blank falls under the third term, multiply the first and second terms for the divisor and the other three for a dividend.

—But—

5. If the blank falls under the first or second term, multiply the third and fourth terms for a divisor, and the other three for a dividend, and the quotient will be the answer.

"Practice" (we are told) "is so called, from the general use thereof by all persons concerned in trade and business."

"All quantities in this rule are performed by taking aliquot or even parts, by which means many tedious reductions are avoided."

Then follows a series of rules for finding the cost of a number of articles at ½d., ¾d., 1½d. (⅛s.), 2d. (⅙s.), 2½d. (⅛s. + ⅓ of 2d.), ... 1/1½d., 1/2½d., ... 1/10½, 1/10⅔, 1/10¾.

A typical rule may be given:

"Rule 3rd.—When the price is more than one shilling and less than two, take the part or parts, with so much of given price as is more than a shilling, which add to the given quantity, and divide by 20, it will give the answer."

Rule 4 is interesting as illustrating a short-cut method.

"When the price consists of any even number of shillings under 20, multiply the given quantity by half the price, doubling the first figure of the product for shillings, and the rest of the product will be pounds."

(*E.g.*) 3123 at 16/-

8

£2498 : 8 Answer.

But, except in this instance, tedious reductions are not generally avoided, and tortuous and wearisome methods prevail. Thus 1/11¾ is separated into 1/-, 6d., 4d., 1d., ½d., ¼d., instead of being conceived as 2/- less ¼d., as most modern text-books advise.

Simple Interest is first defined as "the profit allowed in lending or forbearance of any sum of money for a determined space of time."

Interest is calculated first for one year, then for several. A formula is not used. Principal (or Rate) is found by Double Rule of Three.

Compound Interest is done by continued addition.

"Barter is the exchanging of one commodity for another, and informs the traders so to proportion their goods, that neither may sustain loss."

(*E.g.*) "How many dozens of candles, at 5/2 per dozen, must be delivered in barter for 3 cwt. 2 qr. 10 lbs. of tallow at 37/4 per cwt."

"Profit and Loss is a rule that discovers what is got or lost in the buying or selling goods: and instructs us to rise or fall our price, so as to gain or lose so much per cent., or otherwise."

The examples under these last two headings are worked by "Rule of Three."

Book 2. *January* 1818 *to May* 1818. *Age* 12.

The first 36 pages deal with the conversion of English units of measure into Italian, Portuguese, Dutch, German, Irish.

(*E.g.*) "A gentleman remits to Ireland £575. 15. 0 sterling; what will he receive there, the Exchange at 10 % ?" (Answer £633. 6. 6).

All the Examples are worked by "Rule of Three," or by Conjoined (*i.e.* Compound) Proportion, which is defined thus:

"Conjoined Proportion is when coin, weight, or measure of several countries are compared in the same question, or it is linking together a variety of proportions."

"Arithmetical Progression" (6 pages) comes next, and the work consists in finding each of the quantities s, d, n, a, 1 from three of the others by means of formulae, such as $n=\frac{1-a}{d}+1$, which are first stated verbally.

(*E.g.*) "Divide the fifth by the third, and from the quotient subtract half the product, of the fourth multiplied by the third, less 1 gives the first, or thus, n, d, s, are given to find a.

$$\frac{s}{n}-\frac{dx(n-1)}{2}=a.$$"

Geometrical Progression (5 pages) follows, and is dealt with similarly, except that literal formulae are avoided and the obscure verbal rule remains uncondensed.

(*E.g.*) "In any Geometrical Progression proceeding from unity, the ratio being known to find any remote term without producing all the intermediate terms.

RULE 1. Find what figures of the indices added together would give the exponent of the term wanted; then multiply the numbers standing under such exponent into each other, and it will give the term required."

The examples given are all numerical, and present problems such as that of the bridegroom who promised his bride, as a dowry, one shilling for the first month and "double it on the first day of every month for 1 year," with the result that he had to pay up £20,415 altogether.

There are 10 pages on "Permutation" (*e.g.* "how many changes may be rung upon 12 bells ?"), and the rest of the book is devoted to Vulgar Fractions and Decimals.

Reduction of Vulgar Fractions occupies 35 pages. We give two extracts:

1. "To reduce fractions of one denomination to the fraction of another, but less, retaining the same value.

RULE: Multiply the numerator by the parts contained in the several denominations between it and that you would reduce it to, for a new numerator,

and place it over the given denominator. Reduce the new fraction to its lowest terms.

EXAMPLE. Reduce $\frac{4}{784}$ of a cwt., avoirdupois, to the fraction of a lb.

$$4 \times 28 \times 4 = \frac{448}{784}.$$

G.C.M. (found by repeated division) = 112.

$$112) \frac{448}{784} (\frac{4}{7} \quad \text{Answer."}$$

2. "To reduce a mixed fraction to a single one.

RULE. When the numerator is the integral part, multiply it by the denominator of the fractional part, adding in the numerator of the fractional part for a new numerator; then multiply the denominator of the fraction by the denominator of the fractional part for a new denominator.

EXAMPLE. Reduce $\frac{23\frac{5}{7}}{38}$ to a single fraction.

$$23 \times 7 + 5 = 166 \text{ numerator.}$$
$$38 \times 7 \quad = 266 \text{ denominator.}$$

G.C.M. (found by repeated division) = 2.

$$2) \frac{166}{266} (\frac{83}{133} \quad \text{Answer."}$$

Addition (7 pages), Subtraction (5 pages), Multiplication (7 pages), Division (7 pages), Single Rule of Three Direct in Vulgar Fractions (11 pages), Single Rule of Three Inverse (13 pages) follow in turn.

The remaining 29 pages deal with the first four rules in decimals, and include contracted multiplication and division. The methods are similar to present-day methods, except that standard form is not used. The comma is used as a decimal point instead of a dot.

Book 3. *February* 1819 *to March* 1820. *Age* 13.

This is concerned with "Practical Geometry," and includes the mensuration of surfaces and solids. The syllabus covered is much the same as nowadays. The area of a triangle in terms of three sides, applications of Pythagoras' Theorem, areas of polygons, lengths of arc of circle in terms of chords of arc and half arc, and area of a lune all appear. The solids treated include prism, pyramid, cone, frustums, cunens or wedge, prismoid.

The rule for the prismoid is:

"To the sum of the areas of the two ends add four times the area of the section parallel to and equally distant from both ends, and this last sum multiplied by $\frac{1}{6}$ of the height will give the solidity."

The term "solidity" is used instead of "volume" throughout. Exercises in area do not involve the relation $30\frac{1}{4}$ sq. yards equal one sq. pole. The duodecimal system is used in dealing with areas less than one square foot. Thus, for the area of a rhomboid whose length is 7 ft. 9 ins. and height 3 ft. 6 ins., the working is:

ft.	ins.		
7	9		
3	6		
23	3		
3	10	6	
27	1	6	Answer

But decimals may be used in the first part of a solution. (*E.g.*) Area of square 5 ft. 9 ins. long.

Sq. of $5{\cdot}75 = 33{\cdot}0625$

$$\cdot0625 \times 12 = \cdot75$$
$$\cdot75 \times 12 = 9.$$

Answer. 33 ft. 0 ins. in 9 parts.

This form of answer appears throughout.

The square root sign is written above the number, (*e.g.*)

$$\sqrt{10}, \quad \sqrt{12500}.$$

There is a good deal of elaborate figuring throughout; quantities involving up to a dozen digits are common, the supreme achievement appearing at the end of the book when the area of the earth's surface is calculated as:

198943779·7460695000 square miles,

this answer then being written out correct to 3 decimal places.

Book 4. *December* 1820 *to February* 1821. *Age* 14.

The first half consists of arithmetic applied to the work of bricklayers (2 pages), masons (3), carpenters (4), slaters (3), plasterers (6), painters (6), glaziers (3), paviors (4), and plumbers (3). It deals with the units each uses in measuring the amount or value of work done.

(*E.g.*) "Bricklayers compute or value their work at the rate of a brick and a half thick."

To find the number of rods of brickwork in a wall:

"Multiply the superficial content of the wall in feet, by the number of half bricks in the thickness, and $\frac{1}{2}$ of that product will be the content required."

The examples are all exercises in area or volume, and most of them give further practice in the use of the duodecimal system.

The second half contains a number of rules and problems (16 pages) "of vaulted and arched roofs" (*e.g.*), "spherical dome," "vacuity by an elliptical groin"; then Specific Gravity (11 pages), providing exercises in proportion, and finally "Distances as determined by the velocity of sound."

The last problem is as follows:

"Walking the other day to take the air,
(Bright shone the sun, the weather very fair).
At distance I a dismal cloud did spy,
Which (as I thought) against the wind did fly.
While I upon my watch did look to see,
How time did pass away; Lo! instantly,
A dreadful flash of lightning pierc'd the cloud;
Just fourteen seconds after which aloud
The thunder roar'd: now I inform'd would be,
How many feet the cloud did burst from me?"

	sc.	feet	sc.
As	1	: 1142	:: 14
			14
			4568
		1142	
		15988	feet the Answer.

(This problem is given in at least one text-book of the period which has come into my hands.)

Book 5. *February* 1821 *to* .... *Age* 15.

This book contains "Geometrical Problems," which are practically those in the present School Certificate Course.

There is a short course (5 pages) at the end on Conic Sections, in which the variously shaped sections are diagrammatically illustrated and geometrical constructions for ellipse and parabola are given. Three pages "of solids," in which cube, prism, etc., are described and illustrated, complete the book.

NOTE.

Although it is known that the pupil attended the school 1817-1822, and that he wrote the books, there is no definite evidence that they were written at or for school. At Oswestry Grammar School, however, there are some school record books covering the period 1819-1828, which indicate the work done by each pupil each week, and the arithmetical topics entered are much the same as in the Yorkshire school. R. S. WILLIAMSON.

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532. When a man has got thoroughly puzzled at some passage in an author, or at a mathematical problem, I have known him sit over it for hours, till he was half distracted, without being any the forwarder; and when he comes to look at it again a day or two after, having been occupied in the interim with other things, he finds it quite easy. . . . I always told my pupils, "When, after a reasonable time, you cannot make out a difficulty, pass on to something else, and return to the point next day."—*Life of Whately*, 1868, i. p. 146.

533. Jan. 27th, 1657-8.—After six fits of a quartan ague, our deare son Richard 5 yeares and three days old onely, died . . . he had a wonderful disposition to mathematics, having by heart divers propositions of Euclid that were read to him in play, and he would make lines and demonstrate. . . . *Evelyn's Diary* (Wheatley), ii. 97.

534. Those who think science worth having must read it with what Butler calls the PAIN of attention.

535. He alone discovers who proves.—Paley.

536. Let us find out the thing: there will be no lack of persons to put it into shape.—D'Alembert.

537. (Desaguliers)

> . . . who taught two gracious kings to view
> All Boyle ennobled, and all Newton knew,
> Died in a cell, without a friend to save,
> Without a guinea and without a grave!—Cawthorn.

538. Of Sir Robert Walpole (1727): His reputed mistress, Mrs. Howard, and the Speaker, his reputed Minister, were perceived to be nothing; and Mr. Pulteney and Lord Bolingbroke, in the algebraical phrase, less than nothing.—Croker, *Hervey Memoirs*, 1884, i. p. 52.

539. But though a brilliant mathematician and a skilled chess player, (Mortimer Collins) was a man of the field as well as of the closet, a thoroughly practical botanist and ornithologist, . . .—*N. & Q.* V. 6, p. 200.

540. 3rd August, 1664: a concert of excellent musitians, especialy one Mr. Berkenshaw, that rare artist who invented a mathematical way of composure very extraordinary, true as to the exact rules of art, but without much harmonie.—Evelyn's *Diary*.

541. There is scarcely a single instance of a mathematician of eminence gone mad.

542. [*Below the Salt.* Mrs. Henry Wood.] Edgar is speaking to Pleasaunce: "Now, I kiss you three times on one cheek, and four times on your mouth. How many did that make altogether?" "Seven," whispered the girl, disengaging herself to breathe more freely. "That is arithmetic," said the youth triumphantly. "Dear me," said Pleasaunce, "I should not have thought it." —Cf. Harry's song in Marryatt's *Snarleyyow*.

543. *Scholars*, used of Mathematicians, 1845, Cambridge, "by a singular extension of the term," *v.* Bristed, p. 385.

544. Mathematics fared little better than divinity sixty years ago, study of Euclid, of algebra and even of arithmetic being practically optional at Eton.

MATHEMATICAL NOTES.

886. [V. 8.] *Porson.*

Beloe, *Sexagenarian*, ii. p. 309 (1817), gives the following problem attributed to Porson. "His celebrated equation, given in the former part of your journal, is in every one's hands," writes V. L. in *The Classical Journal*, x. p. 401 (1814). Watson, in *The Life of Richard Porson* (1861), states problem and solution as follows:

Τὶς ὁ ἀριθμός, οὗ τενομένου εἰς δύο ἀνίσους μερίδας, ἡ τῆς μείζονος μερίδος δύναμις μετὰ τῆς ἐλάττονος μεταλαμβανομένη ἴση ἔσεται τῇ τῆς ἐλάττονος δυνάμει μετὰ τῆς μείζονος μεταλαμβανομένῃ.

Required the number, which being divided into two unequal parts, the square of the greater added to the less shall be equal to the square of the less added to the greater.

Let x be the number, and y one of the parts. Then $x-y$ is the other part;

$$\therefore\ (x-y)^2+y=y^2+x-y,$$
$$x(x-2y)=x-2y;$$
$$\therefore\ x=1,$$

showing that practically there are no positive values of the two parts." This solution provokes no comment from the three copyists. Charles Butler, the first Roman Catholic barrister since 1791, tells us that Porson's knowledge of algebra and geometry was considerable, and we know that the great classic meditated a new edition of *Diophantus*. "A short time before he died he gave the Reminiscent an algebraic problem, which, though not of the highest order, is certainly curious." Butler gives the problem, with a solution by "the Reminiscent's learned friend—Mr. Frend." *Reminiscences of Charles Butler, Esq., of Lincoln's Inn*, 1822. William Frend was De Morgan's father-in-law.

The problem was: Solve $xy+zu=444$, $xz+yu=180$; $xu+yz=156$; $xyzu=5184$.

That Porson's knowledge of Geometry was only superficial, says V. L. (*loc. cit.*), "is little better than mere idle report; as is sufficiently evident from the nature of the annexed problem, composed by him, *en capricieux*, as a sort of challenge to the Fellows of Trinity College."

In Euc. I. 47, A is the right angle, and $ACKM$, $ABFL$ are the squares on AC, AB respectively. "The point in which the lines CF, BK intersect is in AL, the perpendicular drawn from the right angle to the base, BC."—R. P.

This he proves by the aid of similar triangles, and a reductio ad absurdum.

887. [A. 2. 2.] *Linear Equations* (*v.* Math. Note, 872, *Gazette* xiii., p. 393).

Consider the system:

$$u_1\equiv a_1x+b_1y+c_1z+d_1=0,$$
$$u_2\equiv a_2x+b_2y+c_2z+d_2=0,$$
$$u_3\equiv a_3x+b_3y+c_3z+d_3=0.$$

Let $\Delta=(a_1b_2c_3)$, and let A_1, B_1, ... be the co-factors of a_1, b_1, ... in Δ; also let

$$\Delta_1=(d_1b_2c_3),\quad \Delta_2=(a_1d_2c_3),\quad \Delta_3=(a_1b_2d_3).$$

Suppose that $\Delta\neq0$, then $(A_1B_2C_3)=\Delta^2\neq0$, therefore at least one term of $(A_1B_2C_3)$ is not zero. Let $A_1B_2C_3\neq0$, so that no one of A_1, B_2, C_3 is zero.*

Because $A_1\neq0$, the equations are equivalent to

$$u_2=0,\quad u_3=0,\quad A_1u_1+A_2u_2+A_3u_3=0;$$

that is, to

$$u_2=0,\quad u_3=0,\quad \Delta x+\Delta_1=0.$$

* It is essential to show that there are suffixes p, q, r, all different, such that none of A_p, B_q, C_r is zero. This is not done in the text-books. (See Chrystal's *Algebra*, vol. i. p. 360.)

In the same way, since $B_3 \neq 0$ and $C_3 \neq 0$, the equations are equivalent to each of the sets:

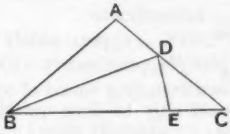
$$u_3=0, \quad u_1=0, \quad \Delta y+\Delta_2=0; \qquad u_1=0, \quad u_2=0, \quad \Delta z+\Delta_3=0.$$

Therefore the equations have the unique solution

$$(-\Delta_1/\Delta, \quad -\Delta_2/\Delta, \quad -\Delta_3/\Delta).$$ S. BARNARD.

888. [K. 1. c.]

ABC is a triangle having the angles B and C each 40°. Prove that, if the bisector of the angle B meets AC at D, $BD+DA=BC$.



Proof. From BC cut off $BE=BD$. Join DE. Thus $\hat{A}=(\pi-\hat{B}-\hat{C})=100°$, and $B\hat{E}D=B\hat{D}E=\frac{1}{2}(\pi-20°)=80°$;

$\therefore$ $BADE$ is a cyclic quad^{l}.,

and $A\hat{B}D=D\hat{B}E$; $\therefore$ the chords AD, DE are equal and $E\hat{D}C=\hat{B}=\hat{C}$;

$$\therefore \ DE=EC; \quad \therefore \ AD=EC.$$

Thus $BD+DA=BE+EC=BC$.

It is easy to deduce that $2\cos 40°$ is a root of $x^3-3x+1=0$. W. F. BEARD.

889. [V. 1. a. δ.] *Note on the Teaching of Logarithms.*

The general practice in teaching logarithms may be fairly described as follows:

(1) It is pointed out that multiplication and division would be very easy, being replaced by addition and subtraction, if we only had to deal with powers of a single number.

(2) A meaning is derived for the power of a number where the index is fractional or negative: a to a fractional index meaning the root of an integral power of a or an integral power of a root of a; and a to a negative index meaning $1 \div a$ to a positive index.

(3) Recourse is then had to a graph (generally of 2^x plotted from integral values of x or of 10^x plotted from values $x=1$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, etc.) to illustrate the evaluation of fractional powers.

(4) The use of tables (with or without an attempt to explain the method of compiling them) is then dealt with, and applications to calculation taken, ending with the statement of the logarithmic laws and notation.

My experience is that step (3) does not adequately fill the gap between step (2) and step (4); for no attempt is made to show how, for example, 2 can be expressed (approximately, of course) as a power of 10, by applying the result of step (2). The appeal to the graph depends merely on getting the pupil to make the assumption, generally unknowingly, that $10^{\cdot3}$ must lie between the plotted values $10^{\cdot25}$ and $10^{\cdot375}$, and that the graph may be used to interpolate such unplotted values without any consideration as to the meaning of such quantities as $10^{\cdot3}$. The acceptance of this assumption must depend more on suggestion than on understanding. The following exposition, to be inserted between steps (3) and (4), is intended to fill in as far as is possible at this stage of the pupil's mathematical knowledge the gap alluded to.

To express 2 as a power of 10.

Since $10^0=1$ and $10^1=10$, we may expect 2 to be 10^x where x is a fraction between 0 and 1.

Is $2=10^{\cdot1}, 10^{\cdot2}, 10^{\cdot3}, 10^{\cdot4}$ ... approximately ?

i.e. to $10^{\frac{1}{10}}, 10^{\frac{2}{10}}, 10^{\frac{3}{10}}, 10^{\frac{4}{10}}$ ... approximately ;

i.e. to $\sqrt[10]{10}, \sqrt[10]{10^2}, \sqrt[10]{10^3}, \sqrt[10]{10^4}$ approximately ;

i.e. $2^{10}=10, 100, 1000, 10{,}000$ approximately.

Now $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$, $2^{10}=2^5\times2^5=32\times32=1024$, which is very nearly 1000 ;

$$\therefore\ 2=10^{\cdot3} \text{ approximately.}$$

To find a better approximation, we proceed similarly to ask :

Is $2=10^{\cdot30}, 10^{\cdot31}, 10^{\cdot32}$, ... approximately ?

i.e. is $2^{100}=10^{30}, 10^{31}, 10^{32}$, ... approximately ?

Now $2^{100}=(2^{10})^{10}=(1000)^{10}$approximately $=10^{30}$ approximately.

(To make this last step satisfactory it would of course be necessary to show that $2^{100}<10^{31}$, which could only be done quickly by using the Binomial Theorem ; but the latter is now generally done long after logarithms.)

The above method of finding logarithms direct from the definition is, of course, too long to be of any use in practice, and this should be pointed out, and an explanation—however vague—entered on as to the existence of more advanced methods of compiling the tables.

Herbert Strutt School, Belper. C. BLACK.

890. [V. 1. a. ϵ] *On Determinants.*

The following proofs of the two important propositions on determinants in §§ 419, 430 of C. Smith's *Algebra* (1910) are somewhat shorter and perhaps a little simpler than those given by the author.

I. Let $P\,.\,h_\alpha\,.\,Q\,.\,k_\beta\,.\,R$ be any term of a determinant where the letters are in alphabetical order, and suppose for definiteness that $\alpha<\beta$. Then if Q contain l suffixes greater than β, m between α and β, and n less than α, moving β to h will cancel l inversions and create n inversions, which moving α to k will restore and destroy respectively. Also both operations will create m inversions. These, together with the inversion in the interchange of α and β will create $2m+1$ inversions, thereby multiplying the term by $(-1)^{2m+1}=-1$.

If $\alpha>\beta$, then obviously the operations are simply reversed.

II. *Multiplication of determinants.*

Consider the determinant

$$[a_1\alpha_1+a_2\beta_1+a_3\gamma_1,\quad b_1\alpha_2+b_2\beta_2+b_3\gamma_2,\quad c_1\alpha_3+c_2\beta_3+c_3\gamma_3].$$

By usual methods (§§ 428, 420, 421) this is equal to

$$\begin{aligned}
&a_1b_2c_3\,[\alpha_1\beta_2\gamma_3]+a_1b_3c_2\,[\alpha_1\gamma_2\beta_3]\\
&+a_2b_3c_1\,[\beta_1\gamma_2\alpha_3]+a_2b_1c_3\,[\beta_1\alpha_2\gamma_3]\\
&+a_3b_1c_2\,[\gamma_1\alpha_2\beta_3]+a_3b_2c_1\,[\gamma_1\beta_2\alpha_3]\\
&=[\alpha_1\beta_2\gamma_3]\,(a_1b_2c_3-a_1b_3c_2+a_2b_3c_1\\
&\qquad -a_2b_1c_3+a_3b_1c_2-a_3b_2c_1)\quad (\S\,420)\\
&=[\alpha_1\beta_2\gamma_3]\,[a_1b_2c_3].
\end{aligned}$$

A. J. CARR.

891. [X. 1.]

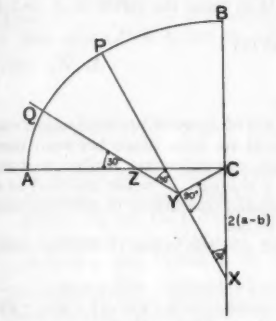
Architects use the following method for drawing a five-centred arch, approximately elliptical : Given semi-axes CA, CB, produce BC to X, making CX equal to $2\,(CA-CB)$. Draw XP at 30° to XB, CY perp. to XP, YQ at 30° to YP, cutting CA at Z. Then X, Y, Z are the centres of 30° arcs forming one side of the arch. Fig. 1 shows this method.

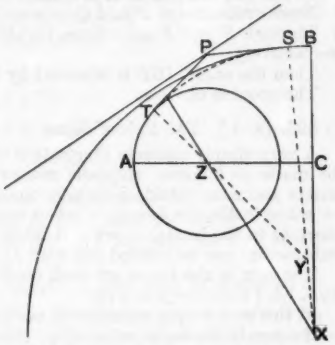
It will be noticed, however, that X is not the centre of curvature of the ellipse at B. Hence the following construction, which gives a closer approximation than the above :

Make BX of length $\frac{CA^2}{CB}$, and AZ (along AC) $\frac{CB^2}{CA}$.

Draw circles, centre X, radius XB, and centre Z, radius ZA.

Construct their radical axis (if XZ cut the circles at F, E respectively, draw equal lines FH, EG, perp. to XZ. Then draw circles radii XH, ZG, and their



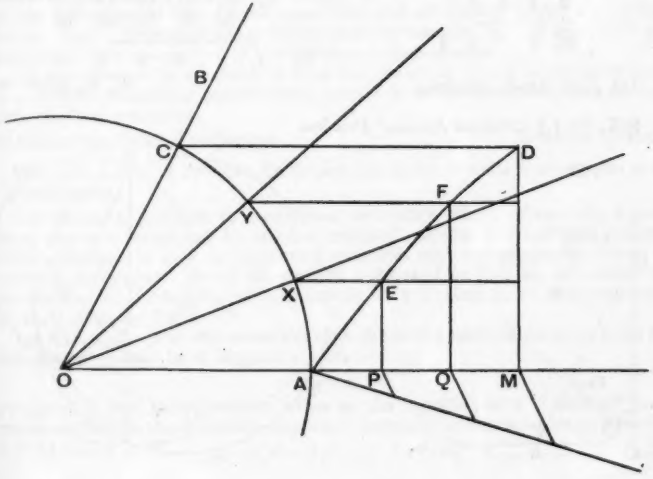


common chord will be the radical axis). From any convenient point P on the radical axis, draw tangents PS, PT to the two circles, and let TZ produced cut SX at Y. Then X, Y, Z are the three centres for one side of the arch.

Felsted School, Essex. E. H. Lockwood.

892. [V. 1. a. ζ] *To divide an angle into any number of equal parts.*

Suppose AOB is the angle. With O as centre and any convenient radius describe a circle, and from this circle construct geometrically a sine graph, taking any convenient scale for the abscissae and letting the x axis coincide with OA.



Suppose that OB cuts the circle at the point C.
Draw CD parallel to OA to meet the sine graph in the point D.
Draw DM perpendicular to OA.
Then AM is a measure of the angle AOB.
Divide AM into the desired number of equal parts, say 3.
Let us call the points of division P and Q.
Erect ordinates at P and Q to meet the sine graph in E and F.
Through E and F draw lines parallel to AO to meet the circle in X and Y respectively.
Then the angle AOB is trisected by OX and OY.
The proof is obvious. A. W. LUCY.

893. [A. 1.] *The Rule of Signs.*

A very simple concrete illustration of the rule of signs in multiplication can be made as follows. Suppose money transactions take place between men inside and men outside a certain house. Take the following units: a man, + when inside the house, − when outside; a shilling, + when credited to a man, or to the house, − when debited. Extended operations of addition and subtraction can be carried out with all these units.

If x men in the house are each credited with y/-, the house is credited with xy/-, *i.e.* $(+x)\times(+y)=+xy$.
In this case x men outside will each owe y/-, *i.e.* $(-x)\times(-y)=+xy$.
If x men in the house each owe y/-, the house owes y/-, *i.e.* $(+x)\times(-y)=-xy$.
In this case x men outside the house are each credited with y/-,

i.e. $(-x)\times(+y)=-xy$. H. V. MALLISON.

894. [V. 1. a. δ.]

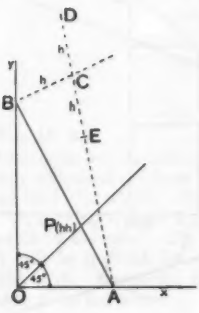
The following methods of multiplication may be interesting to teachers of the elements. The examples explain themselves.

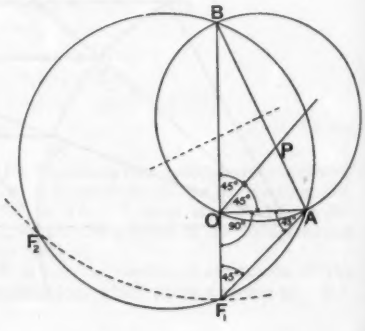
6	7	4	3	2
				4
24	8	6	2	8
2	1	1	0	
26	9	7	2	8

		9	9	7	2	4
				8	3	4
72	72	56	16	32		
	27	27	21	06	12	
		36	36	28	08	16
83	1	6	9	8	1	6

168 Bath Road, Southsea. W. R. MEADOWS.

895. [K. 1.] *Note on Pappus' Problem.*





Pappus' problem consists in constructing a straight line of given length (L), so that its ends shall rest on two given lines OA, OB at right angles, and it shall pass through a given point $P(h, k)$ on the bisector of $B\hat{O}A$. The solution of the problem leads to an equation of the fourth degree which may decompose into two equations of the second degree provided favourable conditions are chosen, *e.g.* suitable choice of unknowns, axes, etc.

Let AB, Fig. 1, be the reqd. line in position and have equation $x/a+y/b=1$.

Then $OA=a$, $OB=b$ and $a^2+b^2=L^2$. .....(1)

Also, since $P(h, k)$ lies on AB;

$$\therefore \frac{h}{a}+\frac{k}{b}=1,$$

i.e. $(a+b)k=ab$. .....(2)

From (1) and (2) we obtain

$$a+b=h\pm\sqrt{h^2+L^2}.$$

This leads to an immediate geometrical construction.

From BC perpendicular to AB cut off $BC=k$; then $AC=\sqrt{h^2+L^2}$.

From AC cut off $CD=CE=h$; then AD and AE give the two values of $(a+b)$, namely, $h\pm\sqrt{h^2+L^2}$.

Referring now to Fig. 2, on $AB=L$ describe a semicircle AOB and an arc of the circle $A\hat{F}_1B$ containing 45°. With centre B and radius the AD of Fig. 1 describe an arc to cut the circle AF_1B in F_1, F_2. Join BF_1 to cut the semicircle AOB in O. Join AO, AF_1, and let the bisector of $A\hat{O}B$ cut AB in P. Then $AOBP$ is a required configuration. For $B\hat{F}_1A=45°$ and $B\hat{O}A=90°$;

$$\therefore O\hat{A}F_1=45°; \quad \therefore OA=OF_1;$$

$$\therefore BO+OA=BF_1=a+b; \quad \therefore BO=b \text{ and } OA=a.$$

From F_2 a second configuration follows.

We might get two other configurations using $a+b=h-\sqrt{h^2+L^2}$, keeping in mind the fact that as $\sqrt{h^2+L^2}>h$, the value of $(a+b)$ would be negative, which means that regarded geometrically we should have to treat $h-\sqrt{h^2+L^2}$ as the difference of two (+)ve quantities and ultimately prefix a (−) sign to the "a." Regarded analytically this corresponds to the case of "a" negative. The completion of the problem is then simple.

All it is desired to establish is that the choice of $(a+b)$ as variable leads to a simple geometrical construction, true it is, of the inverted problem, but, none the less, a solution.

16 Cambridge Road, Teddington. BERTHA NAYLOR.

896. [M¹. 3. 1.] *A formula for the product of the normals from a point to an algebraic curve.*

Sometimes a problem in Analytical Geometry which offers considerable resistance to a treatment by classical methods, admits of an elegant solution when attention is paid to imaginary elements and the projective aspect of metrical invariants. As an illustration I proceed to find an expression for the product of the lengths of the normals from a point (x, y) to an algebraic curve Γ given by $f(x, y)=0$.

Let $\phi(x, y, r)=0$ be the condition that one of the normals from (x, y) to the given curve Γ may be of length r; then

$$\phi(x, y, r)=0$$

admits of a dual interpretation either as the equation of a "parallel" to Γ corresponding to the distance r supposed known in advance, or as an equation in r (or rather in r^2), $A_0+A_1r^2+A_2r^4+\ldots A_nr^{2n}=0$, ...(1)

giving the lengths of the normals from (x, y) to Γ, the A's being polynomials in x, y. If n and m be the order and class of Γ it is known that, in general, a parallel to Γ will be of order $2m+2n$ and that $m+n$ normals can be drawn to Γ from any point in the plane. (Hilton, *Plane Algebraic Curves*, pp. 180 and 163.) Hence ϕ is of order $2m+2n$ in x, y and of order $k=m+n$ in r^2.

The equation $A_k=0$ represents the locus of points from which the length of one or more of the normals is infinite, and as the distance between two points can be infinite only when one of them lies on the Absolute, this locus consists of the Absolute itself, unless the curve Γ has contact with it, in which case $A_k \equiv 0$ and the coefficient of the highest power of r which does not vanish represents the equation of the Absolute. Hence in Euclidean Cartesian Geometry *the coefficient of the highest power of* r *in* (1) *is a constant.*

Again, the equation $A_0=0$ is the locus of points from which one of the normals is of zero length. As the distance between two points vanishes only when the points coincide or when their join has contact with the Absolute, it follows that A_0 vanishes only for points on Γ or on one of the $2m$ normals (*i.e.* tangents) from either of the circular points to the curve. The equation of the latter may, by associating conjugate pairs together, be expressed in the shape

$$h(x, y) = \prod_{i=1}^{m} [(x-\alpha_i)^2 + (y-\beta_i)^2],$$

where (α_i, β_i) are the various *real* foci of the curve. A comparison of the orders of A_0, f and h in x, y which are $2m+2n$, n and $2m$ in the general case, leaves no doubt as to the relation

$$A_0 = f^2 \,.\, h \times \text{a constant.}$$

Perhaps a more convincing argument is that as a point (x, y) approaches the curve $f(x, y)=0$ the length of the infinitesimal normal from it is of the same order of smallness as $f(x, y)$, so that the product of the squares of the normals must be of order $f^2(x, y)$.

From the values of A_0 and A_k just determined it follows that if $n_1 n_2 \dots n_k$ are the lengths of the normals from $P(x, y)$, then

$$n_1 n_2 \dots n_k \equiv f(x, y) \,.\, PS_1 \,.\, PS_2 \dots \times \text{a const.},$$

where $S_1, S_2, \dots$ are the real foci of Γ.

The foregoing analysis cannot be pushed to the extent of determining the constant, but this can in general be done by choosing a special position of P. Thus by taking P at the centre of the ellipse

$$f \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

it is readily seen that *

$$n_1^2 n_2^2 n_3^2 n_4^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2 \,.\, PS_1^2 \,.\, PS_2^2 \,.\, \frac{b^4}{c^4}.$$

If Γ has the line at infinity as a multiple tangent of order μ, then μ of the normals are of infinite length, while an equal number of foci disappear at infinity, but the reasoning still holds for the remaining normals and foci. Thus for the parabola $f \equiv y^2 - 4ax = 0$,

$$n_1^2 n_2^2 n_3^2 = f(x, y) \,.\, PS \times \text{const.}$$
$$= PS^2 \,.\, (y^2 - 4ax)^2,$$

by choosing P at the centre of curvature at the vertex.†

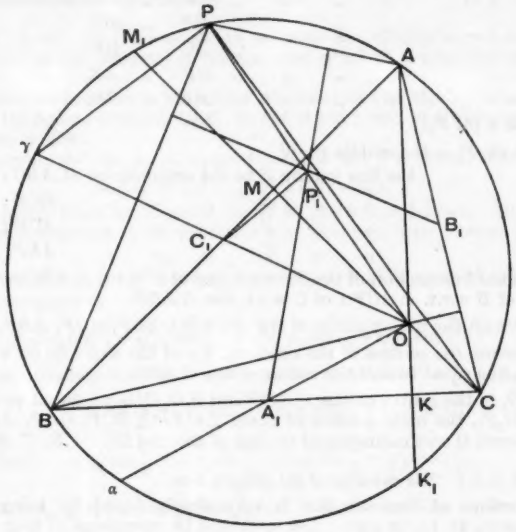
A. NARASINGA RAO.

897. [K'. 8. a.]

1. If $ABCP$ are 4 pts. on a circle, $A_1B_1C_1$ the middle pts. of BC, CA, AB respectively, O the ortho-centre of $\triangle ABC$, and if perpendiculars be drawn

* Salmon, *Conic Sections*, page 338. † *Loc. cit.* page 337.

from $A_1B_1C_1$ $\perp^r$ to AP, BP, CP respectively, these perpendiculars will meet in a pt. P_1 on the circle $A_1B_1C_1$ and P_1 is the middle pt. of OP.



Otherwise, if $A_1B_1C_1P_1$ are 4 pts. on a circle (the nine-pts. $\odot$), $A_1B_1C_1$ being the middle pts. of the sides of the $\triangle ABC$, and if perpendiculars be drawn from $A\,.\,B\,.\,C$ $\perp^r$ to A_1P_1, B_1P_1, C_1P_1 respectively, their $\perp^{rs}$ will meet in a pt. P on the circle ABC.

Start with P_1 the mid-pt. of OP. Let CO produced cut AB in M and the $\odot$ in M_1. Let OC_1, OA_1 meet the circumference in γ, α respectively.

2. $OM = MM_1$ and $OC_1 = C_1\gamma$; $\therefore$ γM_1 is parallel to C_1M ;

$\therefore$ $\angle \gamma M_1 C$ is a right angle ;

$\therefore$ $\angle \gamma PC$ is a right angle, but C_1P_1 is parallel to γP ;

$\therefore$ C_1P_1 is $\perp^r$ to PC.

3. $OK = KK_1$, $OA_1 = A_1\alpha$; $\therefore$ αK_1 is parallel to A_1K ;

$\therefore$ $\alpha K_1 A$ is a right angle,

i.e. $A\alpha$ is a diameter and $\angle APa$ is a right angle.

But A_1P_1 is parallel to αP ;

$\therefore$ A_1P_1 is $\perp^r$ to AP.

Similarly, B_1P_1 can be shown to be $\perp^r$ to BP.

4. The $\perp^r$ from the middle pt. of AP on to BC evidently bisects OP ;

" " " " BP " CA " " "

" " " " CP " AB " " "

5. Hence if 4 pts. $ABCP$ lie on a circle, the 6 st. lines drawn $\perp^{r}$

from the middle pt. of AP on to BC,
" " " " BC " AP,
" " " " BP " AC,
" " " " AC " BP,
" " " " CP " AB,
" " " " AB " CP,

all meet in a pt. P_1.

6. This pt. P_1 is the middle pt. of

the line joining P to the ortho-centre of ABC;
" " A " " " BCP;
" " B " " " ACP;
" " C " " " ABP.

7. P_1 is the intersection of the Simson's line of P w.r.t. ΔABC: of A w.r.t. ΔBCP: of B w.r.t. ΔACP: of C w.r.t. the ΔABP.

8. P_1 lies on the nine-pts. $\odot$s of the Δs ABC, BCP, ACP, ABP.

9. Therefore the centres of the nine-pts. $\odot$s of the 4 Δs lie on a $\odot$ centre P_1 and radius equal to half the radius of the $\odot ABC$.

10. If O_1 is the centre of the $\odot ABC$ and if O_1P_1 is produced to Ω so that $O_1\Omega = 2\,.\,O_1P_1$, the ortho-centres of the 4 Δs ABC, BCP, CAP, ABP lie on a circle centre Ω and radius equal to that of the $\odot ABC$. R. T. ROBINSON.

898. [K. 2. a.] *The envelope of the Simson line.*

The envelope of Simson's line is very simply found by using complex coordinates $(z, \bar{z})$, *i.e.* $(x \pm yi)$. Let α, β, γ, δ be complexes of four points on a unit circle, so that $\alpha\bar{\alpha}=1$, etc., then the equation of $\beta\gamma$ is

$$z+\bar{z}\beta\gamma=\beta+\gamma.$$

The perpendicular through δ is $z-\bar{z}\beta\gamma=\delta-\bar{\delta}\beta\gamma$, meeting the former line, where

$$2z=\beta+\gamma+\delta-\bar{\delta}\beta\gamma,$$

$$i.e.\quad 2z-\alpha-\beta-\gamma-\delta=-\frac{\alpha\delta+\beta\gamma}{\delta},$$

whence

$$2\bar{z}-\bar{\alpha}-\bar{\beta}-\bar{\gamma}-\bar{\delta}=-\frac{\bar{\alpha}\bar{\delta}+\bar{\beta}\bar{\gamma}}{\bar{\delta}}=-\frac{\alpha\delta+\beta\gamma}{\alpha\beta\gamma}=\frac{\delta}{\alpha\beta\gamma}(2z-\alpha-\beta-\gamma-\delta),$$

which incidentally shows that the Simson lines of each point with respect to the triangle formed by the other three are coincident.

To get the envelope for a variable parameter δ, we have, by differentiating for δ, and eliminating $\bar{z}$,

$$\frac{1}{\delta^2}=\frac{1}{\alpha\beta\gamma}(2z-\alpha-\beta-\gamma-2\delta).$$

Transferring the origin to $\frac{1}{2}(\alpha+\beta+\gamma)$, *i.e.* the nine-point centre,

$$z=\delta+\frac{\alpha\beta\gamma}{2\delta^2},$$

which by proper selection of the initial line and initial argument of z, gives an equation of the form $z=e^{\omega i}+\frac{1}{2}e^{-2\omega i}$, *i.e.* a three-cusped hypocycloid.

The cusps are given by $dz=0$, if the corresponding values of arg. δ are real. This gives $\delta^3=\alpha\beta\gamma$, so that arg. $\delta=\frac{1}{3}(\text{arg. }\alpha+\text{arg. }\beta+\text{arg. }\gamma+2n\pi)$. This method has the advantage of indicating the position of the cusps.

The theorem admits of generalisation.

Perpendiculars are drawn to the sides of a triangle at the points where a transversal cuts them, forming a similar triangle. If the magnitude of this triangle is constant (zero in Simson's case), the envelope of the transversal will be a three-cusped hypotrochoid. L. J. ROGERS.

899. [K'. 2. d.] *A trigonometrical solution of Malfatti's problem of describing three circles mutually in contact, each of which touches two sides of a triangle.*

This historical problem is very fully discussed in Coolidge's *Treatise on the Circle and the Sphere* (Oxford, 1916), p. 174 *et seq.*; and in Hobson's *Trigonometry* (1891, p. 209).

In Hobson's notation, ρ_1, ρ_2, ρ_3 are the radii, and it is shown that

$$a=\rho_2\cot\tfrac{1}{2}B+\rho_3\cot\tfrac{1}{2}C+2\sqrt{\rho_2\rho_3}. \quad \text{.....(i)}$$

At this point, however, it seems better to proceed as follows. By taking a second equation similar to (1) and forming an equation homogeneous in the ρ's, we get

$$\sin B(\rho_2\cot\tfrac{1}{2}B+\rho_3\cot\tfrac{1}{2}C+2\sqrt{\rho_2\rho_3})=\sin A(\rho_3\cot\tfrac{1}{2}C+\rho_1\cot\tfrac{1}{2}B+2\sqrt{\rho_3\rho_1})$$

which is equivalent to

$$\left(\sqrt{\rho_2}\cos\frac{B}{2}+\sqrt{\rho_3}\sin\frac{B}{2}\right)^2=\left(\sqrt{\rho_1}\cos\frac{A}{2}+\sqrt{\rho_3}\sin\frac{A}{2}\right)^2. \quad \text{.....(ii)}$$

We must obviously take the positive square root on each side.

Thus, $$\sqrt{\rho_2}\cos\frac{B}{2}+\sqrt{\rho_3}\sin\frac{B}{2}=\sqrt{\rho_1}\cos\frac{A}{2}+\sqrt{\rho_3}\sin\frac{A}{2},$$

while similarly, $$\sqrt{\rho_3}\cos\frac{C}{2}+\sqrt{\rho_1}\sin\frac{C}{2}=\sqrt{\rho_2}\cos\frac{B}{2}+\sqrt{\rho_1}\sin\frac{B}{2}.$$

If we write t_1, t_2, t_3 for $\tan\frac{A}{4}$, $\tan\frac{B}{4}$, $\tan\frac{C}{4}$, these equations reduce to

$$\sqrt{\rho_1}(1+t_1)=\sqrt{\rho_2}(1+t_2)=\sqrt{\rho_3}(1+t_3).$$

From (i) we get after some trigonometrical reductions

$$\rho_1=\tfrac{1}{2}r\frac{(1+t_2)(1+t_3)}{1+t_1}, \text{ etc.} \quad \text{.....(iii)}$$

where r is the radius of the inscribed circle.

An extension of Malfatti's problem may under certain conditions be solved, where the circles are to intersect externally at given angles θ_1, θ_2, θ_3.

For then (i) becomes

$$a=\rho_2\cot\frac{B}{2}+\rho_3\cot\frac{C}{2}+2\sqrt{\rho_2\rho_3}\cos\frac{\theta_1}{2}, \text{ etc.}$$

and (ii) becomes

$$\left(\sqrt{\rho_2}\cos\frac{B}{2}+\sqrt{\rho_3}\sin\frac{B}{2}\cos\frac{\theta_1}{2}\right)^2-\left(\sqrt{\rho_1}\cos\frac{A}{2}+\sqrt{\rho_3}\sin\frac{A}{2}\cos\frac{\theta_2}{2}\right)^2$$
$$=\rho_3\left\{\sin^2\frac{A}{2}\sin^2\frac{\theta_2}{2}-\sin^2\frac{B}{2}\sin^2\frac{\theta_1}{2}\right\},$$

so that if

$$\sin\frac{\theta_1}{2}\Big/\sin\frac{A}{2}=\sin\frac{\theta_2}{2}\Big/\sin\frac{B}{2}=\sin\frac{\theta_3}{2}\Big/\sin C,$$

we get linear equations giving the ratios of the square roots of the radii.

It will be found that $\sqrt{\rho_1}$ is inversely proportional to

$$\left(\cos\frac{B}{2}+\sin\frac{\theta_2}{2}\right)\left(\cos\frac{C}{2}+\cos\frac{\theta_3}{2}\right)-\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}, \text{ etc.}$$

I have not worked out the exact value of ρ_1, corresponding to (iii).

L. J. ROGERS.

900. [B. 4.] *An "Invariant."*

The following invariant is derived as a generalisation of well-known puzzles, examples of which may be found in Ball's *Mathematical Recreations*, pp. 8 to 10.

Let alpha, beta, and gamma be three units such that b gammas = 1 beta and a betas = 1 alpha. Set down x alphas, y betas and z gammas; b, x and z being such that $b > x > z$. From this quantity subtract z alphas, y betas and x gammas as in arithmetic, so:

Alphas.	Betas.	Gammas.
x	$y+a$	$z+b$
$z+1$	$y+1$	x
$x-z-1$	$a-1$	$z+b-x$

Since $x > z$ we must add b gammas to the upper line and compensate by adding one beta to the lower line. Similarly we add a betas to the upper line and one alpha to the lower.

Now reverse the answer to the subtraction and add the line so reversed to the answer, as follows:

Alphas.	Betas.	Gammas.
$x-z-1$	$a-1$	$z+b-x$
$z+b-x+1$	$a-1$	$x-z-1$
b	$a-2$	$b-1$

We write, not $2a-2$, in the beta column, but $a-2$, because a betas = one alpha, and we carry one to the alpha column. Since this result, b alphas, $(a-2)$ betas, $(b-1)$ gammas, is independent of x, y, and z, we may call it an invariant. Now for alpha, beta, gamma we can substitute any three common units from our tables of weights and measures, and appropriate integers for x, y, and z: for instance, take £9 5s. 2d.; in this case $a = 20$, $b = 12$, and $b > x > z$, that is, $12 > 9 > 2$, because were this not so the sum of money obtained on reversing would contain a number of pence greater than twelve, and also the number of pounds in the lower line would be greater than that in the upper.

£		s.		d.
9	,,	5	,,	2
2	,,	5	,,	9
6	,,	19	,,	5
5	,,	19	,,	6
12	,,	18	,,	11

This result is invariable, and it is equivalent to b alphas, $(a-2)$ betas, $b-1$ gammas. The table below gives groups of units which may be operated on as above and also the corresponding invariants:

Alphas.	Betas.	Gammas.	Invariant.		
—	—	—	b	$a-2$	$b-1$
hundreds	tens	units	10	8	9
yards	feet	inches	12 yds.	1 ft.	11 ins.
tons	hundredwts.	quarters	4 tons	18 cwts.	3 qrs.
degrees	minutes	seconds	60°	58′	59″
metres	decimetres	centimetres	10 m.	8 dm.	9 cm.

It is interesting to notice that we here perform, not only the common operation of substituting numbers for variables, but also the uncommon operation, in mathematics, of substituting words for words, *e.g.* yards, feet and inches for alphas, betas and gammas.

JAS. W. STEWART.

901. [V. 1. 9.] In St. John's College Library, from the Todhunter Collection [Press-mark 2.49.57] is:

A System | of | Popular Geometry; | containing | in a few lessons so much of the | Elements of Euclid | as is | necessary and sufficient for a right understanding of | Every Art and Science | in its leading truths and general principles.|

By George Darley, A.B.

The Fifth Edition.

London: Printed for Taylor and Walton, Booksellers and Publishers to the University of London, Upper Gower Street. 1844.

On p. 6. This theorem [I. 4], which is made the fourth proposition in Euclid, should rightly be the first, inasmuch as it is the foundation of Geometry, the corner-stone upon which the whole superstructure of geometrical science rests. Yet viewing it without fear or prejudice, we see how nearly connected with our simplest thoughts and habits, Geometry, the most abstract and refined of all human inventions, really is. The demonstration in Article the first is little more than showing that the *palms of our hands* are equal by putting them together!

Preface is dated June, 1826.

Darley also seems to have written:

2. Companion to the Popular Geometry.
3. A System of Popular Algebra.
4. A System of Popular Trigonometry.
5. Familiar Astronomy.

} Adv. of Darley's Scientific Library on p. iv.

[4. 2nd ed., 1835, is also in the Todhunter Collection.]

T. A. A. Broadbent : F. Puryer White.

902. [V. 1. a. ζ] *Concerning "Subtend."*

I am very much annoyed—boiling over, in fact. I have just returned from a visit to a school where I have heard a miserable schoolboy—in Form IV., I think—ruthlessly hauled over the coals because he did not know the meaning of "subtend." Now I remember that sometime late in my own mathematical career I was in much the same position of ignorance. I believe, if I must be accurate, that it was after coming down from Cambridge that on one occasion I found myself unaware or doubtful of the meaning of "subtend." It is certainly not uncommon to find a schoolboy in this mental plight. But why should a boy be required to know the meaning of this word? Can it even be dignified with the descriptive title "word"? It is merely a geometrical term. At any rate I have never heard it used in an ordinary English sentence apart from geometry, and research in dictionaries has brought to light no instance of its use in a non-geometrical way. It has been allotted a place in an ordinary dictionary, which gives its derivation, viz. Lat. *sub* = under, *tendo* = stretch, incidentally bringing out its Grimm relationship with the term "hypotenuse" (Greek *hupo* = under, *teino* = stretch), and also its meaning, viz. "extend under or be opposite to." So that it *may* eventually be discovered and used in an ordinary way to convey, with philological sanction, the meaning of "stretch under."

But, as far as one can gather, nobody up to the present has ventured to use it thus, and coin some such phrase as, for instance, "he lay in a hammock lazily subtending the blazing sun," or, to try another example, "the horizontal slit which did duty for his mouth subtended a deeply furrowed angle whose apex was eclipsed by a crimson protuberance of aggressive proportions." Perhaps, now that higher education is widespread and so many more are being brought into touch with the fascinating mysteries of geometry and its terminology, some future writer, turning over in his mind intellectual lumber acquired during school days, may show better how this term can be turned to

literary use. But I am concerned with the present, and the present position is that after a very long innings in school "subtend" has failed to find a place in either the standard literature or the current parlance of the day. Why, then, should we continue to impose it on schoolboys, and make the forgetting of its geometrical implication a heinous offence? Surely not in the hope that some day one of our victims will popularise, or at any rate detechnicalise, the term and the idea which it indicates. There are, of course, some grounds for such a hope, since the idea, although not found in any form in the common everyday thought of civilised society, yet exists implicitly in concrete form in many features of our ordinary surroundings. This might, therefore, conceivably be an argument for introducing a few specially gifted boys to the term, but is it a reason for victimising the multitude? Should we not at once banish from our geometrical vocabulary a term which is not essential to the subject, and which is of no practical use to the pupil?

Before we accept this position absolutely, let us look a little further into the matter, for there is a difference between being "used" and being "useful" in practical life, and this distinction is of particular importance in the case of such a theoretic subject as geometry, the educational value of which consists largely in the mental training obtained from the study of it. So let us consider the term and idea "subtend" in relation to the subject in which it occurs.

The geometrical idea "subtend" is a complex one. It includes the idea of "oppositeness," but not that alone. Thus in a triangle ABC, the side BC is opposite the angle A, and we say that BC subtends the angle A; but in the pentagon $ABCDE$, the side CD (or BC or DE) is opposite the angle A, and yet it would be incorrect to say that the side CD subtends the angle A. The idea of "subtend," in fact, includes not only the idea of "oppositeness," but also that of "terminating on the arms of an angle."

Now consider the development of geometry in the mind of a pupil. On arriving at the idea "subtend" in our presentation of the subject, we have two possible ways of dealing with it. On the one hand we may be content to use a word already in the pupil's vocabulary, and to describe the idea by use of the word "opposite." This is a word which is used in several connections in geometry to indicate "non-adjacentness," a connotation which is consistent with and throws light on the ordinary everyday use of the word. Its use in the case under consideration, involving a clear and specific instance of "non-adjacentness," would therefore help to fix and clarify both its geometrical and its everyday meaning, and consequently would have positive educational value.

On the other hand we may enlarge the pupil's vocabulary with the new term "subtend," and make the idea represented by it, which idea is only loosely described by "be opposite to," clear and precise in his mind. In this case it is claimed that he gets certain mental training (*e.g.* in analysis and in precision of thought) which helps in the general development of his mind, and so is *useful* in practical life. It is doubtful, however, whether, at the time "subtend" is usually introduced, the pupil is sufficiently developed to profit by the degree of geometrical precision involved in differentiating between the ideas of "oppositeness" and "subtending." And, even if he were, we do not regard the claim of "mental training" as justified. It might be admitted in case of a new term which was absolutely essential for the appropriate development of the subject. The idea of "alternate angle," for instance, is a fundamental one, and the introduction of this term to describe the idea is justified sufficiently by the part it plays in securing that conciseness of expression which is one of the essential features of geometry. But the discrimination between "subtend" and "be opposite to" is not necessary for the development of geometry, and even if it were, there is not much gain in conciseness in using the former for the latter.

Let us leave these theoretical considerations, however, and look at the argument from the point of view of practical experience. How many teachers do actually make the meaning of "subtend" clear and precise to pupils? The vast majority are content to explain—or perhaps rather, to state—that "subtends" means "is opposite to," without making any attempt to show the exact connotation of the former. And so most pupils never realise this exact meaning, and many do not remember any meaning whatsoever.

Both theoretical and practical considerations then fail to furnish convincing arguments in favour of "subtend."

Is there any strong reason for retaining it?

"What a small thing to make a fuss about!" I can hear some readers ejaculate. "Small thing," 'tis true! An insignificant peg in the complicated structure of geometry! A mere corner in the expansive field of geometrical knowledge! But, after all—

A peg from which discussion may de-pend;
A corner, line of argument subtend.

For it is on account of small items such as this that we are often led back to considerations touching that eternal but fruitful question, "Why are we teaching pupils geometry?"

R. S. WILLIAMSON.

903. [K¹. 21.] *Two close approximations.*

If α, β, γ are the acute angles whose sines are $\frac{47}{50}$, $\frac{19}{20}$, and $\frac{4}{5}$, then (as near as no matter) $\alpha+\beta+\gamma=182^\circ$, and $\alpha-\frac{1}{2}(\beta-\gamma)=47\frac{8}{11}^\circ$. Hence the following constructions, using circles only, obtain (*a*) an arc AP one-ninth of a given circumference, and (*b*) AQ one-eleventh. The errors are less than a millionth of either AP or AQ.

(*a*) On a circle (O) take AB, BC, CD arcs of 60°, so that $ABCD$ is a semicircle; outside the remaining semicircle find E where $AE=DE=AC$, and on (O) mark TT where $ET=EO$. Outside $ABCD$ place F where $AF=CF=EO$; G where $AG=AC$ and $DG=DA$; and H where $AH=TT$ and $DH=EO$. Then on the arc CD find L where $FL=FO$, M where $GM=GO$, and N where $MN=ML$; the point P is given by $HP=HN$.

(*b*) Semicircle $ABCD$ as before, also E and TT, and G; but bring H nearer to B by making $AH=EO$ and $DH=TT$. On (O) take AL an arc of 45° ($EL=OA$); find M near A where $GM=GO$, and N near C where $BN=LM$; then Q is given by $HQ=HN$.

C. E. YOUNGMAN.

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545. Newton was once watching Handel as he played a harpsichord. The only remark he made was on the elasticity of the musician's fingers.

546. . . . To sport an opinion gracefully, to adore Buffon and d'Alembert, to delight in Mathematicks, logick, Geometry and the rule of Right . . . to detest all the charms of Eloquence unless capable of mathematical Demonstration, and more than all to be vigorously incredulous, is to gain the reputation of good sound sense.—W. Beckford, 1776 (from a letter written at the age of seventeen).

547. Cheyne, who with the desire inherent in mathematicians to reduce everything to mathematical images, considers all existence as a cone, allows that the basis is at an infinite distance from the body. And in this distance between finite and infinite, there will be room for ever for an infinite series of indefinable existence.

548. [A very old man, a maltster], seemed to approach the grave as a hyperbolic curve approaches a line—less directly as he got nearer, till it was doubtful if he would ever reach it at all.—From *Far From the Madding Crowd*, Hardy, c. 15. [Per Mr. P. J. Harris, Market Harborough].

REVIEWS.

Theorie der Gruppen von endlicher Ordnung. By A. SPEISER. 2nd Edn. Pp. ix + 251. 15 RM. 1927. (Springer, Berlin.)

The second edition of Professor Speiser's book differs from the first chiefly in the presence of a new chapter, excellently illustrated, on what may be called the theory of wall-paper: as the author remarks, this is probably one of the most ancient branches of mathematics. Apart from this, the treatment of first principles has been somewhat elaborated.

The author has succeeded in the astonishing feat of condensing all the principal propositions of the subject into the space of 250 pages. The style is extremely elegant and concise, with a minimum of comment and illustration. The first half of the book deals with the "abstract" theory of groups: the second with the deeper "representation" theory. It is characteristic of Speiser's point of view that he regards the latter as a branch of the arithmetic of hypercomplex number systems, that is, as an extension into the non-commutative domain of ordinary algebraic-number theory. The representation theory has recently been extended by Weyl and Schur to an important class of continuous groups, and has been laid by the former at the foundation of his version of quantum mechanics, where it had previously found an interesting application in Heisenberg's theory of resonance. No doubt these things are too late to receive mention, but the book provides an admirable introduction to them. The applications actually considered are Galois' theory of equations and crystallography, to each of which a chapter is devoted.

Altogether the author has written a most valuable book which should stimulate interest in this fascinating theory, so rich in contacts not only with other branches of mathematics but also with the latest speculations of natural philosophy.

P. HALL.

Cremona Transformations in Plane and Space. By H. P. HUDSON. Pp. xx + 454. 42s. net. 1927. (Cambridge University Press.)

The theory of Cremona transformations of the plane may be regarded as complete. Noether's theorem, first enunciated by him and by Clifford in 1869 and finally satisfactorily proved by Castelnuovo in 1900, that every such transformation may be resolved into the product of quadratic transformations, forms the coping-stone of the edifice. Miss Hudson then is able to present this part of the subject in a finished form. As she remarks, there are no very obvious gaps in the theory; though of course the consideration of particular cases can be continued as much as one wishes and may lead to results of geometrical interest. And the various problems enumerated by Coble in his interesting report to the American Mathematical Society (*Bull. Amer. Math. Soc.* 28 (1922), 329-64), with reference to the construction of an invariant theory of the Cremona group, and to various applications to algebra and modular functions, will no doubt repay attention. Hence, the account given in the first section of this book, the first with any claims to completeness, is of the greatest possible value.

But with regard to Cremona transformations in three or more dimensions the situation is quite different. Here, again, Miss Hudson's treatment is invaluable, but for a somewhat different reason. The literature of the subject is so voluminous and so widely scattered that this authoritative and well-written presentation of what is already known will prove of incalculable use in furthering progress.

In three dimensions, to which Miss Hudson confines her attention—surely a matter for regret in an up-to-date book—there are two main sources of difficulty. First, the properties of the fundamental system (the base variety) of a homaloidal web, determining the transformation, are no longer simple. In the plane the fundamental system consists of points of assigned multiplicity, connected with the order of the transformation by two simple equations; base-points which are non-ordinary (contacts) can be obtained as particular cases. In space the fundamental system may consist of curves as

well as points, incident with each other in any way; and contact cannot in general be regarded as a limit. The relations which replace the two simple equations of the plane are "ponderous in the simplest cases, and grow the less useful as they are the more general." Also the incidence relations between the fundamental system and what Miss Hudson calls the "principal system," *i.e.* the elements corresponding to the fundamental system of the second space—it is to be hoped that this name will now become standardised; the remarks on p. 14 concerning the "sheer waste of language" at present customary may be heartily endorsed—are of immense variety, Clebsch's theorem for the plane having only partial extensions to space.

The second stumbling-block is the lack of any generalisation to space of Noether's theorem, to which we have referred above. All we know at present is that there is no finite set of fundamental transformations into which any given transformation can be resolved.

But if the general theory is still incomplete a great deal is known about particular cases, and the collection and classification of our knowledge is an important step. Miss Hudson has done this in a way which merits the highest praise. Her book will be an indispensable authority and a source of inspiration to all workers in the field.

The unsolved problems of geometry are perhaps not so famous as those of the theory of numbers, but they certainly exist. One may be mentioned here; it is related to a topic which Miss Hudson discusses, that of involutions. Is the general cubic variety of three dimensions in space of four dimensions a rational variety? The coordinates of its points may be expressed as rational functions of three non-homogeneous parameters, but this does not answer the question. It is known, in fact, that while in a line or in a plane all involutions of sets of points are rational, *i.e.* their sets may be put in (1, 1) correspondence with the points of a space, the similar proposition for three dimensions is false. There are irrational involutions in three dimensions, though not very much is known about them, and the existence of irrational involutions of sets of two points is still an open question, as Miss Hudson remarks. And, according to Segre's article in the *Encyklopädie*, the original problem is unsolved, though Severi has given reasons for thinking that the answer is no. Is it a vain hope that the solution of problems like this is brought appreciably nearer by the publication of this long looked for book? F. P. W.

Invariants of Quadratic Differential Forms. By OSWALD VEBLEN. Pp. viii + 102. 6s. 6d. net. 1927. (Cambridge University Press.)

Professor Veblen's book is not a revision of J. E. Wright's tract with the same title, (Cambridge Tracts No. 9), but a new work. It contains a particularly clear and well-balanced introduction to the calculus of tensors and the theory of differential quadratic forms. The absence of all examples, geometrical or physical, may make it difficult for a reader approaching the subject for the first time to perceive the true "inwardness" of some of the theorems, but it allows the principles and aims of the theory to stand out with rare clearness. The language is everywhere simple, and there is a deliberate avoidance of high-flown abstract definitions. Here is the opening paragraph:

> The theory of differential invariants, like the theory of algebraic invariants, is essentially formal. It has many geometrical and physical applications which have played a large rôle in the development of the theory. Yet, after all, it is the actual formulas which are the essential subject matter of the theory. This, at least, is the point of view we are adopting at present, and so we shall devote a chapter to questions of notation before we try to say what a differential invariant is.

There are two ways in which the notions of affine structure and covariant differentiation are commonly introduced into books on the tensor calculus, one by way of axioms based on geometrical considerations (as in Weyl's *Raum-Zeit-Materie*), the other during the construction of an invariant differential calculus of the greatest possible generality (for example in Schouten's *Ricci-Kalkül*). Professor Veblen follows a third path: in his treatment the definitions emerge

naturally in the course of the search for differential invariants. For example, the "3-index symbols" are first encountered in the proof that if a rational function of the g_{ik} and $\partial g_{ik}/\partial x^r$ is a tensor it is a function of the g_{ik} alone.

The elements of the theory are developed in Chapters II. and III., in which all the well-known formal properties of the tensor R^i_{jrs} turn up in a natural way without interrupting the steady march of the general theory. Chapter IV., devoted to "Euclidean Geometry," is less satisfactory than the others. It is a pity that so many notions belonging to the general tensor calculus—raising and lowering of indices, definition of area, etc.—should appear only in a chapter with this title.

The last two chapters deal with parts of the subject which owe their recent rapid development almost entirely to Princeton mathematicians, led by Prof. Veblen and Prof. Eisenhart. Chapter V. contains a plain and concise account of the Equivalence Problem ("What are the necessary and sufficient conditions that $g_{ik}(x)\,dx^i\,dx^k$ can be changed into a second given form $h_{ik}(\bar{x})\,d\bar{x}^i\,d\bar{x}^k$ by a transformation of co-ordinates?"); it is at the same time an introduction to the methods used by Eisenhart, Veblen, and T. Y. Thomas in their investigations of such questions as the conditions of integrability of the equations

$$g_{ik,\,r} = 0$$

when the Γ's are given and the g's are the unknowns, (the comma denoting covariant differentiation). Chapter VI. deals with Riemann's normal co-ordinates, neglected for many years but revived by American mathematicians. If P_0 is a point of an n-dimensional manifold and (x) any co-ordinate system, the co-ordinates of a point P, near P_0, in the *normal system* associated with the point P_0 and the system (x), are $u^i s$, where s is the length of the geodesic arc PP_0, and u^i are the components, in the system (x), of its direction at P_0. The k*th affine extension* of a $(p:q)$-tensor field T is defined to be the object whose components in a given system (x) have at P_0 the values

$$\left(\frac{\partial^k T^*}{\partial y^{i_1}\,\partial y^{i_2}\ldots\partial y^{i_k}}\right) \text{ taken at } P_0$$

where (y) is the *normal* system associated with the system (x) and the point P_0, and T^* stands for a y-component of T. It is shown that this object is a $(p:q+k)$-tensor: that it is not a mere combination of covariant derivatives (except when $k=1$) is proved by the fact that in a Riemannian manifold the "affine extensions" of g_{ik} after the first do not in general vanish, whereas the covariant derivatives of g_{ik} of all orders are, of course, zero. The affine extensions are used to establish the fundamental theorem on the expressions of rational tensor functions of g_{ik} and its partial derivatives as rational functions of R_{ijkl} and its covariant derivatives.

At the end of each chapter is an excellent historical summary and short bibliography.

I have found only two minor matters for complaint. First, it seems a pity to omit all reference to the method of infinitesimal changes of co-ordinates. That this is not without "importance for applications"—the author's own criterion—is shown by the use Weyl has made of it to prove that the first variation, $\mathfrak{A}^{ik}$, of any invariant integral which is a function of g_{ik} and its derivatives alone, satisfies the identity $\mathfrak{A}^{ik}_{\ \ k} = 0$, which plays the part of the "equations of energy and momentum" in the corresponding field theory. An extensive account of this side of the theory would no doubt have impaired the unity of Prof. Veblen's treatment, but even a simple reference to the chapter on "Lie's Methods" in Wright's tract would have reduced the danger of a reader's entirely overlooking this side of the subject.

Secondly, a point of notation. The symbols used are of course those employed by Princeton mathematicians, which are generally very convenient and sensible: in particular the temptation, which seems to assail every writer on the subject, to alter the notation at one of those few points where all other writers are agreed, is avoided. It is all the more regrettable that Prof. Veblen did not see his way to adopt what is probably the most important improvement in the notation of the Tensor Calculus since Einstein's summation con-

vention—I mean the round and square brackets of Bach and Schouten, denoting the symmetrical and skew parts of a tensor. For example :

$$a_{(ijk)} = \frac{1}{3!}(a_{ijk} + a_{jik} + a_{jki} + a_{kji} + a_{kij} + a_{ikj}),$$

$$a^{[ijk]}_{r} = \frac{1}{3!}(a^{ijk}_{r} - a^{jik}_{r} + a^{jki}_{r} - a^{kji}_{r} + a^{kij}_{r} - a^{ikj}_{r}),$$

$$a^{(i}b^{j)k} = \frac{1}{2!}(a^{i}b^{jk} + a^{j}b^{ik}),$$

and so on. (Bianchi's identity is $R^{i}_{j[rs,t]} = 0$.) These symbols lighten the formulae very greatly ; they lead to convenient rules of manipulation (*e.g.* if X stands for any row of letters

$$a_{X}b^{(X)} = a_{(X)}b^{X} = a_{(X)}b^{(X)},\ a_{(X)}b^{[X]} = a_{[X]}b^{(X)} = 0, \text{ etc.});$$

and they bring out clearly the importance of the symmetry properties of tensors in all parts of the subject. It is a great pity they have found so little general favour. Prof. Veblen uses the " Kronecker δ's " in place of the square bracket,—*e.g.* he writes

$$\delta^{ijk}_{rst}a^{rst} \text{ for } a^{[ijk]},$$

—and he denotes $a^{(ijk)}$ by $P(a^{ijk})$. He has no notation for $a^{(ij)k}$ or more complicated functions.

These regrets are trifling compared with the pleasure I have derived from reading Prof. Veblen's lucid and easily flowing pages. The book may be warmly recommended, not only to pure mathematicians seeking an introduction to the subject, but also to the physicist who has a little knowledge of Einstein's theory—say, from reading the first three chapters of Eddington's treatise. He will find that nothing is required of him but a knowledge of the elements of the differential calculus and the ability to follow a mathematical argument, and he will return to the study of relativity with a very much clearer perception of what is " behind " the tensor analysis. The only pity is that the title of the book, although the only possible one, will frighten away many physicists who would derive much benefit from reading it.

M. H. A. NEWMAN.

Foundations of Euclidean Geometry. By H. G. FORDER. Pp. xii + 349 25s. net. 1927. (Cam. Univ. Press.)

Opinions may differ as to the value of writing a book with the main object of establishing securely that particular system of geometry known as Euclidean ; to the pure geometer, to employ all the resources of mathematical logic, and in the end to leave him with the straight edge and compasses as his chief weapons is like earthquakes bringing forth mice. But Mr. Forder makes out a very strong case, and his best argument is the very thorough book he has produced. The book provides, as it sets out to do, " a connected and rigorous " —even ' vigorous ' as on the dust-cover—" account . . . in the light of modern investigations." If there is not much that is actually new (although the author is perhaps over-meticulous in his acknowledgments), we have here for the first time in English a full investigation of the logical bases of Euclid. For the student the book supplies first-rate practice in axiom-chopping, with all the advantages of working towards a known goal. The author's solicitude for the reader's logical welfare leads him sometimes into giving unnecessary demonstrations of theorems, as when he shows how to deduce " $A, B \rightarrow A, B$ " ($\rightarrow$ being an arbitrary relationship between the ordered couples of symbols) from " if $A, B \rightarrow C, D$, and $C, D \rightarrow E, F$, then $A, B \rightarrow E, F$ " and " $A, B \rightarrow B, A$," and it is extremely probable that, in spite of the warning in the preface, most readers will skip the rigid demonstrations of theorems of ordinary " elementary " geometry with which the book is liberally sprinkled. Particularly helpful are the chatty comments, in smaller type, which describe the implications of the axioms and theorems, or the purpose of the work about to be undertaken.

The geometric part of the book opens with the consideration of eight axioms of order, which assign the significance of " between " for points, and limit

the geometry to a three-dimensional one, in which lines which lie in the plane of a triangle and meet one of the side-intervals, meet one other side-interval, and on the basis of these, the ideas of regions and angles are discussed. But the system is of course very vague until the "axioms of congruence" are added. In these the properties of the relation of congruence between intervals are stated, and two further restrictions on the geometry are imposed, the first being equivalent to Euclid I. 5 in a weakened form and the second dealing with congruent angles. With these axioms a definition of right angles is obtained (another axiom has yet to be added to ensure that all right angles are congruent), and of angle bisectors, angular measure, etc. During this discussion the question of theoretical and practical constructibility arises, and a minimum of possible constructions is postulated in order that all constructions required may be effected. For example, the order axioms lead to "Any two distinct points define a line uniquely," it is now postulated that "Any two distinct points can be joined by a line." There are five of these constructions required by the axioms of order and congruence (one of them is at first very strong, involving the possibility of transference of intervals, but it is weakened to a more acceptable form subsequently), and two others are added with the later axioms. Circles are next added to the system, and various forms of the axiom about their intersections are considered. The geometry is now well on the way to becoming Euclidean, little more than some sort of parallel axiom being required. The weakest form possible for the axiom is obtained, and then (skipping a number of "elementary" theorems) the reduction of the Euclidean to a projective geometry is effected, and a system of axioms for projective geometry is given.

Having obtained the geometry by the steady formulation of axioms as they are wanted, the next process is to find its properties and see which of these could be substituted for axioms to give a simpler system. To this end coordinates are defined, and the properties of the algebraic and geometric systems compared by the Euclidean analogue of the more familiar projective theory. The chapter on coordinates is rounded off by a discussion of the nature of the dependence of Pappus' and Desargues' theorems on the axioms. The axiom-chopping process is brought to a triumphant conclusion with the introduction of a Dedekind axiom, and it is shown that a sufficient basis for Euclidean geometry is the Dedekind axiom and the axioms of order, together with either a strong parallel axiom, or the weakest possible parallel axiom and the axioms of congruence so far as they relate to intervals. The sequence of the argument is interrupted just before this by a very lucid account of the analysis situs of plane polygons, of the theory of their areas, and the volumes of polyhedra. At the end of the book there are useful chapters on an alternative set of axioms (that due to Pieri), and the theory of areas in Non-euclidean Geometry.

T. G. Room.

Treatise on Thermodynamics. By Dr. Max Planck. 3rd Edn. Translated from the seventh German edition by Alexander Ogg. Pp. xiv + 297. 15s. net. 1927. (Longmans.)

Every student of thermodynamics who must read it in English will be grateful to Professor Ogg for making an up-to-date edition of Planck's *Thermodynamics* available to him.

Planck's *Thermodynamics* is the standard text-book which develops the theoretical consequences of the first and second laws of thermodynamics, and deduces from them the conditions for thermal equilibrium in heterogeneous systems with any number of constituents, treating gas mixtures and dilute solutions in detail. It covers the whole of molar thermodynamics except thermo-electricity and the thermodynamics of radiation. The argument is everywhere clear and easy to grasp, and each step follows naturally from that which precedes it. This is especially true of the deduction of the existence of entropy from the impossibility of a perpetual motion of the second kind, which is first carried out for perfect gases and then extended to other substances. In the eighth German edition, recently published, Dr. Planck has replaced this deduction by that due to Carathéodory. The latter, treating all substances at the same time, and extracting the maximum of conclusions

from the minimum of premisses, certainly surpasses the former in logical beauty, but, in my opinion, would be easy to grasp only to one already familiar with the former or with some other scaffolding on which the latter could be based.

The new edition, thirty-five pages longer than the first English edition, contains the new and important final chapter on Nernst's theorem. The translation is uniformly good, though in one or two places inclined to be too literal; 'by' (p. 114, line 1) should be 'for,' and I noticed one or two other like errors. The printing also is good; the dozen or so misprints I noticed were mostly trivial omissions of suffixes, etc., in formulae; 'T_0' (p. 6, line 14) should be 'θ_0' and 'T' in the last formula on page 123 should be 'T^2'; on the latter page ψ should be replaced by Ψ in nine places. There is an adequate index, but references to original papers are given only in footnotes. It is a pity that a work that has become a classic should be disfigured by even these small faults.

Thermodynamics applied to Engineering. By A. F. Macconochie. Pp. xiv + 260. With illustrations and 2 folding charts. 12s. 6d. net. 1927. (Longmans.)

The aim of this book is stated in its preface to be "to present the principles of engineering thermodynamics in the simplest fashion, and to illuminate these conceptions by reference to the best British and American practice in the major fields of their application," and its standard "that which is characteristic of University degree requirements on both sides of the Atlantic."

The first section, of fifty odd pages, contains a bare outline of general thermodynamics, as far as suffices to explain the Mollier chart and obtain Clapeyron's equation, and a more detailed discussion of various cycles, followed by ten pages on the motion of fluids in nozzles. The second section starts with the thermodynamics of the boiler, the advantage of a second working substance, mercury, being discussed; the theory of the steam engine and the steam turbine follows. The third section contains the theory of the internal combustion engine, the Diesel engine and the gas-turbine being discussed in detail, and finishes with twelve pages on the theory of refrigeration. Each section concludes with a list of "topics for discussion," some worked examples, and a set of examples. Tables of the thermodynamic properties of steam, ammonia, mercury vapour, and carbon dioxide, are provided, as well as Mollier charts for steam and ammonia.

The best part of the book is the clear theoretical discussion with a useful amount of detail of particular machines of different types; the presentation of principles, however, seems in part too sketchy and in part too detailed. The space given to the gas-turbine is surely disproportionately large.

L. H. Thomas.

Algebraic Arithmetic. By E. T. Bell. American Mathematical Society Colloquium Publications, Volume VII. Pp. 1-180. $2·50. 1927. (New York.)

This book deals with a thesis on the border lines of mathematics and logic, a region which has been extensively explored by the American school of mathematics during the last twenty years. Since the appearance of the great works of Whitehead and Russell on the principles of mathematics, which set many new trains of thought agoing on both sides of the Atlantic, there has been, particularly in America, a notable development of the basic principles of algebra. Among several experts, one of whom is the author of the book under review, the names of E. H. Moore and L. E. Dickson of Chicago may be mentioned for their special influence in this movement. While Moore has developed the more logical side, Dickson has developed the mathematical technique, and, speaking broadly, Dickson has enabled us to proceed from a vague notion of *algebra* to a very explicit idea of *an algebra*, very much as nowadays we find it convenient to distinguish in geometry between one geometry such as that of Euclid and another such as that of Bolyai.

If we may at once adopt this point of view—that *an* algebra exists, and therefore presumably several distinct species of algebra exist—then ordinary algebra, as the schoolboy knows it, is *an* algebra, and ordinary arithmetic,

based as it is on the fundamental laws of ordinary algebra, $a+b=b+a$, etc., is now classed as the particular arithmetic of this particular algebra. This is perhaps rather like putting the cart before the horse; for most of us think of arithmetic as the more fundamental, and of algebra as proceeding from arithmetic. But the recent work of Dickson certainly suggests that this reversal may be important, and undoubtedly he is producing several brands of arithmetic, each tallying with one species of algebra. The earlier work of Dickson appears in a Cambridge Tract 16, *Linear Algebras*, and in the first edition of *Algebras and their Arithmetics*. A German translation of this, which was recently published under the author's guidance, contains many significant alterations and simplifications.

Now some familiarity with the methods and terminology of Dickson's work is presupposed in the mind of the reader who takes up *Algebraic Arithmetic*, so that the book cannot form an introduction to this really fascinating subject without some extra help from without. Perhaps this is a little unfortunate, because in many respects the book is very suggestive. It is just the kind that is wanted in these days of specializing. It enables, let us say, the algebraist to learn something of the theory of number or of multiply periodic functions—and conversely. Also the author has broad views on the whole range of algebra; nor is he afraid of giving personal opinions, in delicately worded sentences, somewhat after the manner of Chasles or Klein, which throw a flood of light on the subject. There are, for instance, two very pleasing remarks on the last page (p. 176): one commits the author to the genial view that the theory of logical structure is indeed a department of arithmetic; and presumably this means that to become master of his own craft the logician must now sit at the feet of the mathematician. A very wholesome occupation! Of course arithmetic here means something exceedingly abstract. It is named *rational arithmetic*, and its characteristic feature is that in it are elements bearing a resemblance to the natural numbers, and that these elements may undergo a process called factorization, and that this process for each given element is *unique*.

The other pleasing remark is the very last sentence, wherein after voyaging through strange seas of thought in company with our author, we finally are told that "the ultimate difficulties of arithmetic reside in the natural numbers rather than in their extensions." So all these new vistas of number laws, these quaternions of Hamilton, these q numbers of Dirac, these generalized matrices where the elements of rows and columns cannot even be put into array because they outnumber the ordinal numbers, and these doctrinal functions of Moore—all these are not to be looked on as the playthings of mathematicians who have grown tired of the ordinary numbers 1, 2, 3, but rather as a measure of their desire to begin to learn something about the positive integers!

It appears that when the researches of Dedekind and Kronecker revealed new types of arithmetic where factorization was not unique—non-Euclidean arithmetic indeed—then this became known as *general arithmetic*. A dangerous epithet: for now we learn that "the injection of Dickson arithmetics into the theory of numbers" has thrown all into the melting pot again. Still we have that final sentence of the book to console us: the real business is with the natural numbers.

Probably anyone interested in pure mathematics will find some special appeal in this book. An instance occurs in the handling of a chapter on umbral symbols—a highly technical device invented by Cayley in the invariant theory. Very curious results are here rapidly worked out by putting these umbrae in relation with the exponential function; and incidentally they throw light on quite elementary trigonometrical formulae. Then follows a typical remark, suggesting that as ordinary umbral symbols are connected with simple power series such as the binomial or exponential series, so any more elaborate series generating Legendre or Bessel functions and the like would also have its own umbrae. Extremely simple equalities between umbrae frequently appear as the origin or the core of complicated analytic or algebraic relations involving these functions, or Bernoullian and Eulerian numbers. Why this is so the author does not say, but probably it is only a picturesque restatement

of the familiar fact that such functions satisfy linear differential equations of the type $f(\delta, x)\, y = 0$ where f is a *polynomial* in δ, x; and δ is a differential operator. Symbolic methods of working with such operators provide an instance of umbral symbols, and the fact that we have this *instance* of the behaviour of objects partially analogous to numbers is a strong reason for examining the abstract question which their behaviour suggests.

Chapter I. is so condensed that it is unnecessarily difficult, partly because of its queer arrangement and partly for its unfamiliarity. Certain postulates are stated, on which all depends, but they stand overleaf interspersed with comments and after a page of discussion. This is the part of the book which chiefly needs an introductory knowledge of the technical terms in the works of Dickson already mentioned.

A stimulating little book, very nicely printed. H. W. TURNBULL.

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549. No part of a tangent can go in the circle because it has to come like a pyramid behind. [From a recent "Script."]

550. William Adams, born at Gillingham, near Chatham, was pilot-major, in 1598, of a fleet of five ships which sailed for the East from Rotterdam. The fleet passed through the straits of Magellan and was scattered; the Charity, in which Adams was, ultimately arrived in 1600 on the coast of Japan. Adams was taken before the Emperor and consigned to prison, but eventually rose to favour "by reason I learned him some points of geometry and understanding of the art of mathematics, with other things which pleased him, so that what I said he would not contrary it. . . ."—*Letters received by the East India Company*, vol. 1 (1896), p. 149 [per Mr. F. Puryer White].

551.

Here Einstein lies;
At least, they laid his bier
Just hereabouts—
Or relatively near.

Kensal Green, *Premature Epitaphs, Mostly written in Malice.*

552. Your problem of the Quadrilateral [Gleaning 403] suggests the similar one of the House Agent or College Bursar's description of the estate equidistant from three villages at given distances apart, and so nearly in a line that the estates would be off the island.—Letter of Sir G. Greenhill.

553. "Nauticus" remarks that the article of *Geometricus, N. & Q.* II. vi. 373, "has demolished the vicious reasoning of Professor De Morgan in a masterly style," and enquires why De Morgan "has never dared to reply," suggesting it is "because he has made the discovery that any attempt to controvert the meaning of his opponent would necessarily result in proving *Geometricus* to be the better logician."—III. viii. 75.

554. I vow and affirm your tailor must needs be an expert geometrician; he has the longitude, latitude, altitude, profundity, every dimension of your body so exquisitely . . . as if your tailor were deep read in astrology, and had taken measure of your honourable body with a Jacob's staff, an ephimerides.—Massinger's *Fatal Dowry*, IV. i.

555. In 1849 the Sandhurst textbooks included: Scott's *Elements of Arithmetic and Algebra*, Scott's *Mensuration and Trigonometry*, Narrien's *Euclid, Conic Sections and Practical Astronomy*, Doucharlat's *Differential and Integral Calculus and Dynamics.*

About this time there was a mathematical master—still remembered [1900] by some Senior officers of the Army—who considered it a part of his duty to send every day one of his pupils to the cells.—Mockler-Ferryman, *Annals of Sandhurst*, pp. 37-8.

MATHEMATICAL ASSOCIATION.

YORKSHIRE BRANCH.

THE Spring Meeting of the Yorkshire Branch of the Mathematical Association was held in the Mathematical Department of the University of Leeds on Saturday, 11th February, 1928. Seven new members were proposed, and thirty-six people were present. Mr. W. Peaker, B.Sc., of Cockburn High School, Leeds, was in the chair. Professor E. A. Milne, M.A., M.Sc., F.R.S., Beyer Professor of Applied Mathematics in the University of Manchester, gave a most instructive and interesting paper on "The Radiative Equilibrium of a Rotating Star." Dr. H. Lowery, M.Sc., M.Ed.F.Inst.P., Head of the Physics Department, Technical College, Huddersfield, initiated in a very able manner a discussion on Mr. Montagnon's paper given at the autumn meeting on "The Teaching of Mechanics in Schools." A lively discussion followed in which Misses Abbott, Bowman, Greenwood, Messrs. Clayborn, Montagnon, Bishop, and Professors Brodetsky and W. P. Milne all took part. A committee on the subject was formed on the motion of Professor S. Brodetsky, M.A., Ph.D., of Leeds University.

ARTHUR B. OLDFIELD.

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556. Johnson said that a man had repeatedly called on him, refusing to leave his name. When they did meet, the man confessed that he was oppressed with scruples of conscience. Explanation led to Johnson saying: "tease me no more about such airy nothings," and he was going to be very angry when it occurred to him that the man might be mad, so he asked him how his day was spent, and found he had five hours unemployed. "Five hours of the twenty-four are enough for a man to go mad in; so I would advise you, sir, to study algebra, if you are not an adept already in it: your head would get less muddy. . . . When Johnson felt his own fancy disordered he fled to arithmetic, but probably thought this man had enough arithmetic in the counting-house.

557. What arithmetic will serve to fix the proportion between good and bad wives, and rate the different classes of each? Sir Isaac Newton himself, who could measure the courses of the planets, and weigh the earth as in a pair of scales—even he had not algebra enough to reduce that amiable part of our species to a just proportion; and they are the only heavenly bodies whose orbits are yet uncertain.—Hume, 1751 (Hill's *Hume and Strahan*, p. 347, n. 5).

558. [Sir George Lyttleton] was made, in 1754, Cofferer and Privy-counsellor. This place he exchanged next year for the great office of Chancellor of the Exchequer; an office, however, that required some qualifications which he soon perceived himself to want. . . . It is an anecdote no less remarkable than true that Lord Lyttleton never could comprehend the commonest rules of Arithmetick.—Nichols, *Literary Anecdotes*, 1812, vi. 462.

559. Mr. [John] Whiteside [F.R.S. 1718] is going to London to visit the Virtuosos, being also a Fellow of the Royal Society. . . . He has lately had a Collegium-Mathematicum of a month's course, where he taught the youth of several colleges, particularly the young and most hopeful studious Duke of Queensbery, Marquis of Hartington, etc. He goes through all the parts of the Natural Philosophic Experiments, and Mathematics, for which he is very well accomplished, with excellent instruments well made, at an expence of near 300 *l.* His price is a guinea and a half.—Dr. Charlett (Master of Univ. Coll. Oxford) to Dr. Richardson, Lady Day, 1716. John Nichols, *Illustrations of the Literary History of the 18th Century*, 1817, vol. i. p. 295.

560. George Lyttleton opened the Budget well enough in general, but was strangely bewildered with figures; he stumbled over millions, but dwelt purposely on farthings.—H. Walpole, Jan. 24, 1756.

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THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), the North-Eastern District (Newcastle-upon-Tyne), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by MESSRS. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.